Triangle of Velocities
and
Mathematical Invalidity of the Lorentz Transformation

Aleksandar Vukelja
aleksandar@masstheory.org

www.masstheory.org

November 2005.

Abstract

Lorentz Transformation is mathematically incorrect set of equations. This article presents the general case proof of invalidity, independent of derivation procedure.

Author presents new solution which is named Triangle of Velocities. It is just an example of a mathematically correct setup where expressions similar in form to Lorentz transformation are valid.

The previous, April 2005 version of this article, which is focused on Einstein's own terribly flawed derivation dated 1920, can be downloaded from http://www.masstheory.org/lorentz.pdf

1. Triangle of Velocities

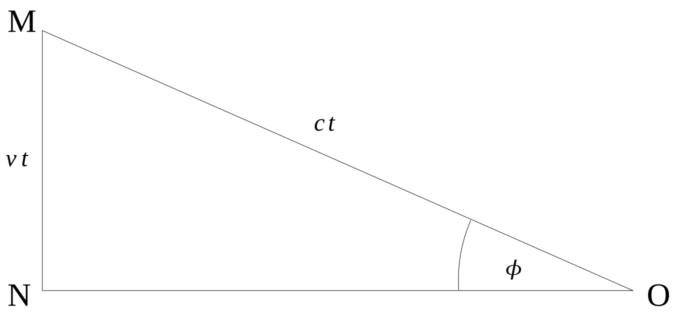
The linear equation set which is usually associated with transformation of coordinates,

$$\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned} \qquad (1.1)$$

has very simple mathematical solution, and we will derive it now. The key to solution is that these equations do not contain information on how are x and x' axes oriented relative to one another.

Let us examine right triangle which is defined with MN = vt and MO = ct.

Distance MN is traveled by a material point at speed v, and distance MO is traveled by a ray of light at speed c in the same amount of time t.



Angle ϕ is defined then with $\sin\phi = \frac{vt}{ct} = \frac{v}{c}$. Because of trigonometrical identity $\sin^2\phi + \cos^2\phi = 1$ we have

$$\cos\phi = \sqrt{1 - \frac{v^2}{c^2}} \qquad (1.2)$$

If we mark length NO with x', MO with x, and $\cos\phi = \beta$, we have

$$x' = \beta x \qquad (1.3)$$

If material point would travel along NO and along MO with the same speed v, the times t' and t required for x' = NO and x = MO would be naturally different because of different lengths of adjacent leg and hypotenuse.

Substituting $x' = vt'$ and $x = vt$ we have

$$t' = \beta t = \beta^{-1}\beta^2 t$$

$$t' = \beta^{-1}\left(1 - \frac{v^2}{c^2}\right) t$$

$$t' = \frac{t - \frac{v^2}{c^2} t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and finally,
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.4}$$

We can use the same procedure for x' now

$$x' = \beta x = \beta^{-1} \beta^2 x$$
$$x' = \beta^{-1} (1 - \frac{v^2}{c^2}) x$$
$$x' = \frac{x - \frac{v^2}{c^2} x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And finally,
$$x' = \frac{x - \frac{v^3}{c^2} t}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.5}$$

Expressions (1.4) and (1.5) are valid mathematical solution for (1.1) equation set in general case. Note that (1.4) and (1.5) are not transformation for pairs (x,t) and (x',t') in which both x and t or x' and t' are arbitrary; it is required that $x = vt$, and $x' = vt'$. Otherwise right triangle would be undefined.

1.1. Examples

We can show now on a couple of examples what expressions (1.4) and (1.5) mean.

Example 1. Triangle of velocities is defined with $v = 0.866c$. What is the angle between adjacent leg and hypotenuse?

We have seen in (1.2) that cosine of that angle is $\cos\phi = \sqrt{1 - \frac{v^2}{c^2}} = 0.5$

Therefore $\phi = \arccos 0.5 = 60^0$

Example 2. If in triangle of velocities, defined with $v = 0.866c$, it takes 1 second to travel full length of adjacent leg, how long would it take to travel the full length of hypotenuse with the same speed v?
From (1.3) we know $t' = \beta t$ and also $\beta = \cos\phi = 0.5$.
Therefore, $t = \beta^{-1} t' = 2 sec$.

2. Mathematical Invalidity of Lorentz Transformation

2.1 Derivation of the Lorentz transformation

The Lorentz transformation is always derived for two coordinate systems *K* and *K'* in relative uniform motion, with clocks reset to zero as they pass by one another.

From standpoint of mathematics, there is no reason that clocks must be reset when *K* and *K'* coincide; we will now derive Lorentz transformation in a more general form which allows that clocks are reset when *K* and *K'* are at any distance.

We start with assumption that transformation of coordinates must be linear:

$$\begin{aligned} x' &= Ax+Bt \\ t' &= Cx+Dt \end{aligned} \tag{2.1}$$

Clocks are reset to zero $t = t' = 0$ when *K* and *K'* are at distance x_0. Therefore we have

$$\begin{aligned} x' &= A(x-x_0)+Bt \\ t' &= C(x-x_0)+Dt \end{aligned} \tag{2.2}$$

First equation can be written as

$$x' = A(x-x_0-(-\frac{B}{A})t) \tag{2.3}$$

For all events at origin of *K'* we have $x' = 0$ and $x = x_0 + vt$. By substituting this in (2.3) we find that speed of *K'* relative to *K* is $v=-\frac{B}{A}$, and (2.3) becomes:

$$x' = A(x-x_0-vt) \tag{2.4}$$

There is a symmetry in that speed *v* of *K'* relative to *K* must be equal to speed of *K* relative to *K'*, but of opposite sign. Writing this in differential form we have:

$$v=-\frac{B}{A}=-\frac{dx'}{dt'}=-\frac{d(A(x-x_0)+Bt)}{d(C(x-x_0)+Dt)} \tag{2.5}$$

For all events at origin of K, we have $x = 0$, so (2.5) becomes

$$\frac{B}{A}=\frac{B}{D} \quad \text{or} \quad A=D \tag{2.6}$$

Next, we can write second equation from (2.2) in the following form:

$$t' = A(t+E(x-x_0)) \tag{2.7}$$

where $E = C/A$.

The transformation must be valid for all events traveling at the speed of light relative to origin of K':

$$x = x_0 + ct \tag{2.8}$$

$$x' = ct' \tag{2.9}$$

Notice that in (2.8) we introduce "invariant speed c", speed which is the same relative to both moving systems. Normally we would have written $x = x_0 + (c + v)t$. We will return to this later.

From (2.4), (2.7), (2.8) and (2.9) we find $E = -\frac{v}{c^2}$.

Now our transformation becomes:

$$\begin{aligned} x' &= A(x - x_0 - vt) \\ t' &= A\left(t - \frac{v}{c^2}(x - x_0)\right) \end{aligned} \tag{2.10}$$

Inverse of (2.10) for $x_0 = 0$ is:

$$\begin{aligned} x &= A(x' + vt') \\ t &= A\left(t' + \frac{v}{c^2}x'\right) \end{aligned} \tag{2.11}$$

After substituting[1] (2.11) into (2.10) we find $A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Finally we have solution

$$x' = \frac{x - x_0 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{v}{c^2}(x - x_0)}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.12}$$

Notice how for $x_0 = 0$ this very simply reduces to the well know form, derived for the case when clocks are reset as the K and K' coincide.

There are also some other derivation procedures, but they happen to be variations in style only, not in essence, as can be seen in various literature and on the Internet.

We can now demonstrate that the highlighted equation (2.8) is mathematical error and that consequential "Lorentz transformation" (2.12) is mathematical error itself.

---

1 For $x_0 \neq 0$ inverse of 2.10 substituted back into 2.10 resolves into identity. Thus constant A can be determined this way only for $x_0 = 0$

2.2 Invalidity of Lorentz Transformation

Lorentz transformation is ambiguous when the same events are observed from different coordinate systems.

This ambiguity is already presented in Elementary Concepts of Material World, but it can be accurately described with Lorentz transformation in the form presented with equation set (2.12).

We will analyze the most basic case of relative uniform motion between two systems, *K* and *K'*. Clocks in *K* and *K'* are reset when *K'* is at distance *X* as observed from *K*.

Observed from K: When *K'* reaches *K*, using second equation of (2.12), we have $x = 0$, $X = -vt$, therefore

$$t' = t\sqrt{1-\frac{v^2}{c^2}} \quad \text{or shortly} \quad t' = \beta(v)t \qquad (2.13)$$

Observed from K': There is no distinction between the two systems. Observed from *K'*, *K* is approaching with speed *v*, and as the two systems coincide, the same equations apply, only variables switch their places:

$$t = t'\sqrt{1-\frac{v^2}{c^2}} \quad \text{or} \quad t = \beta(v)t' \qquad (2.14)$$

From (2.13) and (2.14) we have

$$\beta^2(v) = 1 \qquad (2.15)$$

and this can only be true for limit case when *c* grows to infinity, $c \rightarrow \infty$. In other words, *c* is not invariant finite number. It was an error to assume that it is in (2.8).

When $c \rightarrow \infty$ the Lorentz transformation becomes Galilean:

$$\begin{aligned} x' &= x - x_0 - vt \\ t' &= t \end{aligned} \qquad (2.16)$$

If we did not assume invariant *c*, and had written (2.8) as $x = x_0 + (c + v)t$, the expressions following (2.8) would quickly reduce to Galilean transformation.

As a conclusion, we can say that “Lorentz transformation” is nothing more than a mathematical error.