Chapter II

SPECIAL RELATIVITY

There are limitations on motion that are missed by the Galilean description. The first limitation we discover is the existence of a maximal speed in nature. The maximum speed implies many fascinating results: it leads to observer-varying time and length intervals, to an intimate relation between mass and energy, and to the existence of event horizons. We explore them now.

5. MAXIMUM SPEED, OBSERVERS AT REST, AND MOTION OF LIGHT

> Fama nihil est celerius.*

Light is indispensable for a precise description of motion. To check whether a line or a path of motion is straight, we must look along it. In other words, we use light to define straightness. How do we decide whether a plane is flat? We look across it,** again using light. How do we measure length to high precision? With light. How do we measure time to high precision? With light: once it was light from the Sun that was used; nowadays it is light from caesium atoms. Page 1170

In other words, light is important because it is the standard for *undisturbed motion*. Physics would have evolved much more rapidly if, at some earlier time, light propagation had been recognized as the ideal example of motion.

But is light really a phenomenon of motion? This was already known in ancient Greece, from a simple daily phenomenon, the *shadow*. Shadows prove that light is a moving entity, emanating from the light source, and moving in straight lines.*** The obvious conclusion that light takes a certain amount of time to travel from the source to the surface

* 'Nothing is faster than rumour.' This common sentence is a simplified version of Virgil's phrase: *fama, malum qua non aliud velocius ullum.* 'Rumour, the evil faster than all.' From the *Aeneid*, book IV, verses 173 and 174.

** Note that looking along the plane from all sides is not sufficient for this: a surface that a light beam touches right along its length in *all* directions does not need to be flat. Can you give an example? One needs other methods to check flatness with light. Can you specify one? Challenge 565 n

*** Whenever a source produces shadows, the emitted entities are called *rays* or *radiation*. Apart from light, other examples of radiation discovered through shadows were *infrared rays* and *ultraviolet rays*, which emanate from most light sources together with visible light, and *cathode rays*, which were found to be to the motion of a new particle, the *electron*. Shadows also led to the discovery of *X-rays*, which again turned out to be a version of light, with high frequency. *Channel rays* were also discovered via their shadows; they turn out to be travelling ionized atoms. The three types of radioactivity, namely *α-rays* (helium nuclei), *β-rays*

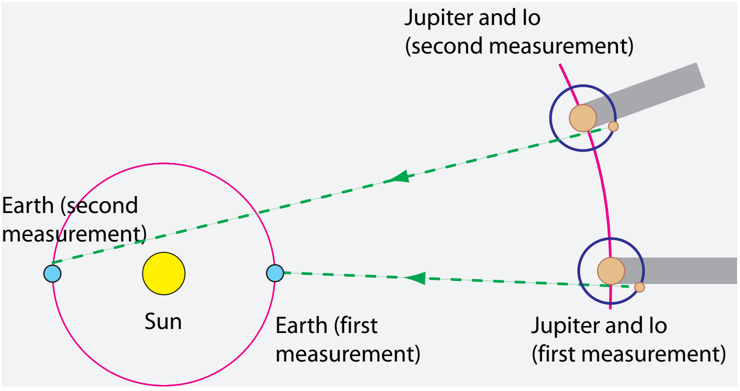

FIGURE 145 Rømer's method of measuring the speed of light

Ref. 246 showing the shadow had already been reached by the Greek thinker Empedocles (*c.* 490 to *c.* 430 BCE).

We can confirm this result with a different, equally simple, but subtle argument. Speed can be measured. Therefore the *perfect* speed, which is used as the implicit measurement standard, must have a finite value. An infinite velocity standard would not allow measurements at all. In nature, the lightest entities move with the highest speed. Light, which is indeed light, is an obvious candidate for motion with perfect but finite speed. We will confirm this in a minute. Challenge 566 n

A finite speed of light means that whatever we see is a message from the *past*. When we see the stars, the Sun or a loved one, we always see an image of the past. In a sense, nature prevents us from enjoying the present – we must therefore learn to enjoy the past.

The speed of light is high; therefore it was not measured until 1676, even though many, including Galileo, had tried to do so earlier. The first measurement method was worked out by the Danish astronomer Ole Rømer* when he was studying the orbits of Io and the other Galilean satellites of Jupiter. He obtained an incorrect value for the speed of light because he used the wrong value for their distance from Earth. However, this was quickly corrected by his peers, including Newton himself. You might try to deduce his method from Figure 145. Since that time it has been known that light takes a bit more than 8 minutes to travel from the Sun to the Earth. This was confirmed in a beautiful way fifty years later, in 1726, by the astronomer James Bradley. Being English, Bradley thought of the 'rain method' to measure the speed of light. Page 140 Challenge 567 n Page 103 Ref. 247

How can we measure the speed of falling rain? We walk rapidly with an umbrella, measure the angle α at which the rain appears to fall, and then measure our own velocity

(again electrons), and γ-*rays* (high-energy X-rays) also produce shadows. All these discoveries were made between 1890 and 1910: those were the 'ray days' of physics.

* Ole (Olaf) Rømer (1644 Aarhus – 1710 Copenhagen), Danish astronomer. He was the teacher of the Dauphin in Paris, at the time of Louis XIV. The idea of measuring the speed of light in this way was due to the Italian astronomer Givanni Cassini, whose assistant Rømer had been. Rømer continued his measurements until 1681, when Rømer had to leave France, like all protestants (such as Christiaan Huygens), so that his work was interrupted. Back in Denmark, a fire destroyed all his measurement notes. As a result, he was not able to continue improving the precision of his method. Later he became an important administrator and reformer of the Danish state.

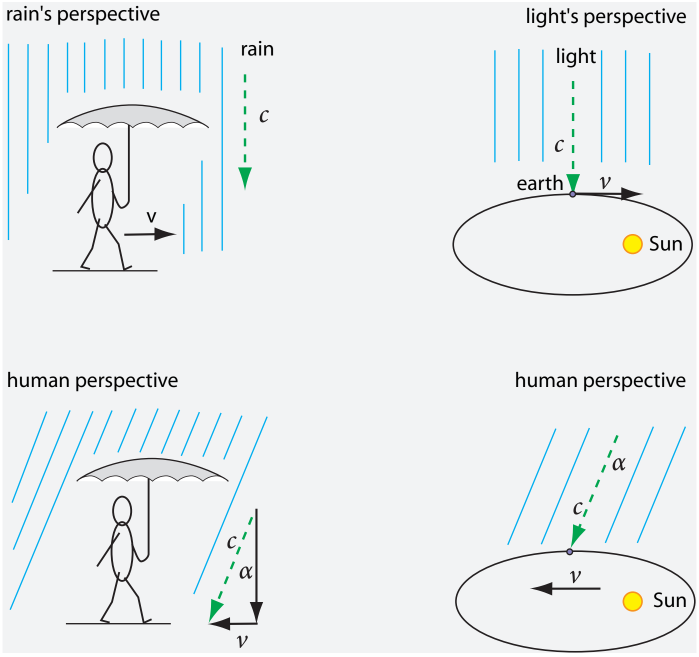

FIGURE 146 The rain method of measuring the speed of light

v. As shown in Figure 146, the speed c of the rain is then given by

$$c = v/\tan\alpha \ . \tag{101}$$

The same measurement can be made for light; we just need to measure the angle at which the light from a star above Earth's orbit arrives at the Earth. Because the Earth is moving relative to the Sun and thus to the star, the angle is not a right one. This effect is called the *aberration* of light; the angle is found most easily by comparing measurements made six months apart. The value of the angle is $20.5''$; nowadays it can be measured with a precision of five decimal digits. Given that the speed of the Earth around the Sun is $v = 2\pi R/T = 29.7\,\mathrm{km/s}$, the speed of light must therefore be $c = 3.00 \cdot 10^8\,\mathrm{m/s}$.* This is

* Umbrellas were not common in Britain in 1726; they became fashionable later, after being introduced from China. The umbrella part of the story is made up. In reality, Bradley had his idea while sailing on the Thames, when he noted that on a moving ship the apparent wind has a different direction from that on land. He had observed 50 stars for many years, notably Gamma Draconis, and during that time he had been puzzled by the *sign* of the aberration, which was *opposite* to the effect he was looking for, namely the star parallax. Both the parallax and the aberration for a star above the ecliptic make them describe a small ellipse in the course
Challenge 568 n of an Earth year, though with different rotation senses. Can you see why?

By the way, it follows from special relativity that the formula (101) is wrong, and that the correct formula
Challenge 569 n is $c = v/\sin\alpha$; can you see why?

To determine the speed of the Earth, we first have to determine its distance from the Sun. The simplest method is the one by the Greek thinker Aristarchos of Samos (*c.* 310 to *c.* 230 BCE). We measure the angle between the Moon and the Sun at the moment when the Moon is precisely half full. The cosine of that angle gives the ratio between the distance to the Moon (determined, for example, by the methods of page 119) and
Challenge 570 n the distance to the Sun. The explanation is left as a puzzle for the reader.

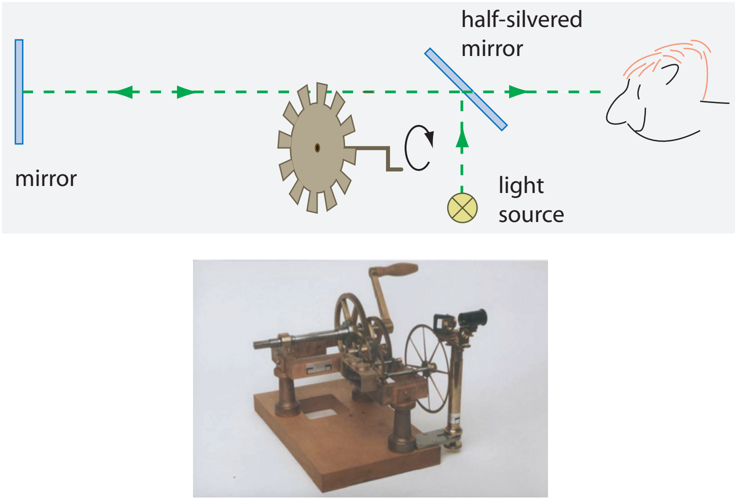

FIGURE 147 Fizeau's set-up to measure the speed of light (© AG Didaktik und Geschichte der Physik, Universität Oldenburg)

an astonishing value, especially when compared with the highest speed ever achieved by a man-made object, namely the Voyager satellites, which travel at 52 Mm/h = 14 km/s, with the growth of children, about 3 nm/s, or with the growth of stalagmites in caves, about 0.3 pm/s. We begin to realize why measurement of the speed of light is a science in its own right.

The first *precise* measurement of the speed of light was made in 1849 by the French physicist Hippolyte Fizeau (1819–1896). His value was only 5 % greater than the modern one. He sent a beam of light towards a distant mirror and measured the time the light took to come back. How did Fizeau measure the time without any electric device? In fact, Page 58 he used the same ideas that are used to measure bullet speeds; part of the answer is given Challenge 572 n in Figure 147. (How far away does the mirror have to be?) A modern reconstruction of Ref. 250 his experiment by Jan Frercks has achieved a precision of 2 %. Today, the experiment is much simpler; in the chapter on electrodynamics we will discover how to measure the Page 570 speed of light using two standard UNIX or Linux computers connected by a cable.

The speed of light is so high that it is even difficult to prove that it is *finite*. Perhaps the most beautiful way to prove this is to photograph a light pulse flying across one's field of view, in the same way as one can photograph a car driving by or a bullet flying through

---

The angle in question is almost a right angle (which would yield an infinite distance), and good instru- Ref. 248 ments are needed to measure it with precision, as Hipparchos noted in an extensive discussion of the problem around 130 BCE. Precise measurement of the angle became possible only in the late seventeenth century, when it was found to be 89.86°, giving a distance ratio of about 400. Today, thanks to radar measurements Page 1183 of planets, the distance to the Sun is known with the incredible precision of 30 metres. Moon distance vari- Challenge 571 n ations can even be measured to the nearest centimetre; can you guess how this is achieved?

Ref. 249 Aristarchos also determined the radius of the Sun and of the Moon as multiples of those of the Earth. Aristarchos was a remarkable thinker: he was the first to propose the heliocentric system, and perhaps the first to propose that stars were other, faraway suns. For these ideas, several of his contemporaries proposed that he should be condemned to death for impiety. When the Polish monk and astronomer Nicolaus Copernicus (1473–1543) again proposed the heliocentric system two thousand years later, he did not mention Aristarchos, even though he got the idea from him.

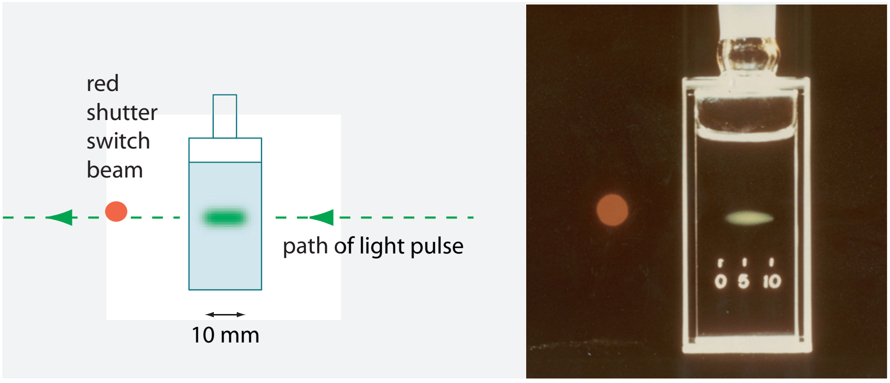

FIGURE 148 A photograph of a light pulse moving from right to left through a bottle with milky water, marked in millimetres (photograph © Tom Mattick)

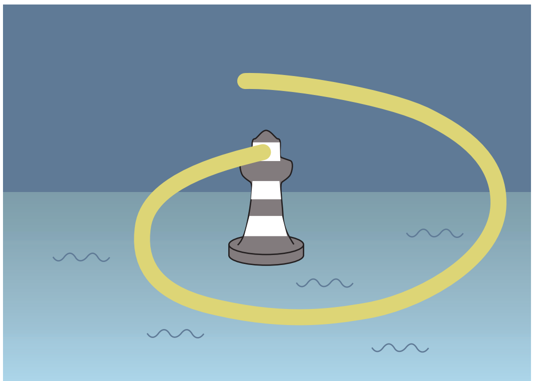

FIGURE 149 A consequence of the finiteness of the speed of light

Ref. 251 the air. Figure 148 shows the first such photograph, produced in 1971 with a standard off-the-shelf reflex camera, a very fast shutter invented by the photographers, and, most noteworthy, not a single piece of electronic equipment. (How fast does such a shutter have
Challenge 573 n to be? How would you build such a shutter? And how would you make sure it opened at the right instant?)

A finite speed of light also implies that a rapidly rotating light beam behaves as shown as in Figure 149. In everyday life, the high speed of light and the slow rotation of lighthouses make the effect barely noticeable.

In short, light moves extremely rapidly. It is much faster than lightning, as you might
Challenge 574 n like to check yourself. A century of increasingly precise measurements of the speed have culminated in the modern value

$$c = 299\,792\,458\ \mathrm{m/s}. \qquad (102)$$

In fact, this value has now been fixed *exactly*, by definition, and the metre has been defined in terms of c. Table 35 gives a summary of what is known today about the motion of light. Two surprising properties were discovered in the late nineteenth century. They form the

TABLE 35 Properties of the motion of light

Observations about light
Light can move through vacuum.
Light transports energy.
Light has momentum: it can hit bodies.
Light has angular momentum: it can rotate bodies.
Light moves across other light undisturbed.
Light in vacuum always moves faster than any material body does.
The speed of light, its true signal speed, is the forerunner speed. Page 593
In vacuum its value is 299 792 458 m/s.
The proper speed of light is infinite. Page 305
Shadows can move without any speed limit.
Light moves in a straight line when far from matter.
High-intensity light is a wave.
Light beams are approximations when the wavelength is neglected.
In matter, both the forerunner speed and the energy speed of light are lower than in vacuum.
In matter, the group velocity of light pulses can be zero, positive, negative or infinite.

Ref. 252 basis of special relativity.

Can one play tennis using a laser pulse as the ball and mirrors as rackets?

> Et nihil est celerius annis.*
>
> Ovid, *Metamorphoses*.

We all know that in order to throw a stone as far as possible, we run as we throw it; we know instinctively that in that case the stone's speed with respect to the ground is higher. However, to the initial astonishment of everybody, experiments show that light emitted from a moving lamp has the same speed as light emitted from a resting one. Light (in vacuum) is never faster than light; all light beams have the same speed. Many specially designed experiments have confirmed this result to high precision. (Ref. 253) The speed of light can be measured with a precision of better than 1 m/s; but even for lamp speeds of more than 290 000 000 m/s no differences have been found. (Can you guess what lamps were used?) (Challenge 575 n)

In everyday life, we know that a stone arrives more rapidly if we run towards it. Again, for light no difference has been measured. All experiments show that the velocity of light has the *same value* for all observers, even if they are moving with respect to each other or with respect to the light source. The speed of light is indeed the ideal, perfect measurement standard.**

* 'Nothing is faster than the years.' Book X, verse 520.

** An equivalent alternative term for the speed of light is 'radar speed' or 'radio speed'; we will see below why this is the case. Page 574

The speed of light is also not far from the speed of neutrinos. This was shown most spectacularly by the

Albert Einstein

Ref. 256 There is also a second set of experimental evidence for the constancy of the speed of light. Every electromagnetic device, such as an electric toothbrush, shows that the speed of light is Page 544 constant. We will discover that magnetic fields would not result from electric currents, as they do every day in every motor and in every loudspeaker, if the speed of light were not constant. This was actually how the constancy was first deduced, by several researchers. Only after understanding this, did the German–Swiss physicist Albert Einstein* show that the constancy is also in agreement with the motion of bodies, as we will do in this section. The connection between electric toothbrushes and re- Page 544 lativity will be described in the chapter on electrodynamics.(For information about direct influences of relativity on machine Ref. 258 design, see the interesting textbook by Van Bladel.) In simple terms, if the speed of light were not constant, observers would be able to move at the speed of light. Since light is a wave, such observers would see a wave standing still. However, electromagnetism forbids the such a phenomenon. Therefore, observers cannot reach the speed of light.

In summary, the velocity v of any physical system in nature (i.e., any localized mass or energy) is bound by

$$v \leqslant c \,. \tag{103}$$

This relation is the basis of special relativity; in fact, the full theory of special relativity is contained in it. Einstein often regretted that the theory was called 'Relativitätstheorie' or 'theory of relativity'; he preferred the name 'Invarianztheorie' or 'theory of invariance', Ref. 259 but was not able to change the name.

---

observation of a supernova in 1987, when the flash and the neutrino pulse arrived a 12 seconds apart. (It is not known whether the difference is due to speed differences or to a different starting point of the two flashes.) What is the first digit for which the two speed values could differ, knowing that the supernova was Challenge 576 n $1.7 \cdot 10^5$ light years away?

Experiments also show that the speed of light is the same in all directions of space, to at least 21 digits of Ref. 254 precision. Other data, taken from gamma ray bursts, show that the speed of light is independent of frequency, Ref. 255 to at least 20 digits of precision.

* Albert Einstein (b. 1879 Ulm, d. 1955 Princeton); one of the greatest physicists ever. He published three important papers in 1905, one about Brownian motion, one about special relativity, and one about the idea of light quanta. Each paper was worth a Nobel Prize, but he was awarded the prize only for the last one. Also in 1905, he proved the famous formula $E_0 = mc^2$ (published in early 1906), possibly triggered by an idea of Page 326 Olinto De Pretto. Although Einstein was one of the founders of quantum theory, he later turned against it. His famous discussions with his friend Niels Bohr nevertheless helped to clarify the field in its most counter-intuitive aspects. He explained the Einstein–de Haas effect which proves that magnetism is due to motion inside materials. In 1915 and 1916, he published his highest achievement: the general theory of relativity, one of the most beautiful and remarkable works of science.

Being Jewish and famous, Einstein was a favourite target of attacks and discrimination by the National Socialist movement; in 1933 he emigrated to the USA. He was not only a great physicist, but also a great Ref. 257 thinker; his collection of thoughts about topics outside physics are worth reading.

Anyone interested in emulating Einstein should know that he published many papers, and that many of them were wrong; he would then correct the results in subsequent papers, and then do so again. This happened so frequently that he made fun of himself about it. Einstein realizes the famous definition of a genius as a person who makes the largest possible number of mistakes in the shortest possible time.

The constancy of the speed of light is in complete contrast with Galilean mechanics, and proves that the latter is *wrong* at high velocities. At low velocities the description remains good, because the error is small. But if we want a description valid at *all* velocities, we have to discard Galilean mechanics. For example, when we play tennis we use the fact that by hitting the ball in the right way, we can increase or decrease its speed. But with light this is impossible. Even if we take an aeroplane and fly after a light beam, it still moves away with the same speed. Light does not behave like cars. If we accelerate a bus we are driving, the cars on the other side of the road pass by with higher and higher speeds. For light, this is *not* so: light always passes by with the *same* speed.*

Why is this result almost unbelievable, even though the measurements show it unambiguously? Take two observers O and Ω (pronounced 'omega') moving with relative velocity v, such as two cars on opposite sides of the street. Imagine that at the moment they pass each other, a light flash is emitted by a lamp in O. The light flash moves through positions $x(t)$ for O and through positions $\xi(\tau)$ (pronounced 'xi of tau') for Ω. Since the speed of light is the same for both, we have

$$\frac{x}{t} = c = \frac{\xi}{\tau} \; . \qquad (104)$$

However, in the situation described, we obviously have $x \neq \xi$. In other words, the constancy of the speed of light implies that $t \neq \tau$, i.e. that *time is different for observers moving relative to each other*. Time is thus not unique. This surprising result, which has been confirmed by many experiments, was first stated clearly in 1905 by Albert Einstein. Though many others knew about the invariance of c, only the young Einstein had the courage to say that time is observer-dependent, and to face the consequences. Let us do so as well.

Challenge 577 e

Ref. 260

Already in 1895, the discussion of viewpoint invariance had been called the *theory of relativity* by Henri Poincaré.** Einstein called the description of motion without gravity the theory of *special relativity*, and the description of motion with gravity the theory of *general relativity*. Both fields are full of fascinating and counter-intuitive results. In particular, they show that everyday Galilean physics is wrong at high speeds.

Ref. 256

The speed of light is a limit speed. We stress that we are not talking of the situation where a particle moves faster than the speed of light *in matter*, but still slower than the speed of light *in vacuum*. Moving faster than the speed of light in matter is possible. If the particle is charged, this situation gives rise to the so-called *Čerenkov radiation*. It corresponds to the V-shaped wave created by a motor boat on the sea or the cone-shaped shock wave around an aeroplane moving faster than the speed of sound. Čerenkov radiation is regularly observed; for example it is the cause of the blue glow of the water in nuclear reactors. Incidentally, the speed of light in matter can be quite low: in the centre of the Sun,

* Indeed, even with the current measurement precision of $2 \cdot 10^{-13}$, we cannot discern any changes of the speed of light with the speed of the observer.

Ref. 254

** Henri Poincaré (1854–1912), important French mathematician and physicist. Poincaré was one of the most productive men of his time, advancing relativity, quantum theory, and many parts of mathematics.

The most beautiful and simple introduction to relativity is still that given by Albert Einstein himself, for example in *Über die spezielle und allgemeine Relativitätstheorie*, Vieweg, 1997, or in *The Meaning of Relativity*, Methuen, London, 1951. It has taken a century for books almost as beautiful to appear, such as the text by Taylor and Wheeler.

Ref. 261, Ref. 262

the speed of light is estimated to be only around 10 km/year, and even in the laboratory, for some materials, it has been found to be as low as 0.3 m/s. In the following, when we use the term 'speed of light', we mean the speed of light in vacuum. The speed of light in air in smaller than that in vacuum only by a fraction of a percent, so that in most cases, the difference can be neglected.

Ref. 263, Ref. 264

Special relativity in a few lines

The speed of light is constant for all observers. We can thus deduce all relations between what two different observers measure with the help of Figure 150. It shows two observers moving with constant speed against each other in space-time, with the first sending a light flash to the second, from where it is reflected back to the first. Since light speed is constant, light is the only way to compare time and space coordinates for two distant observers. Two distant clocks (like two distant metre bars) can only be compared, or synchronized, using light or radio flashes. Since light speed is constant, light paths are parallel in such diagrams.

Ref. 265

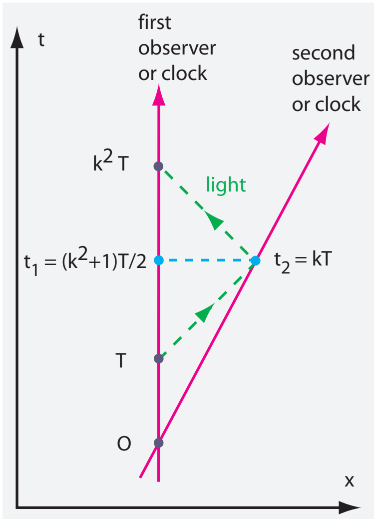

FIGURE 150 A drawing containing most of special relativity

A constant relative speed between two observers implies that a constant factor k relates the time coordinates of events. (Why is the relation linear?) If a flash starts at a time T as measured for the first observer, it arrives at the second at time kT, and then back again at the first at time k^2T. The drawing shows that

Challenge 578 n

Challenge 579 n

$$k = \sqrt{\frac{c+v}{c-v}} \quad \text{or} \quad \frac{v}{c} = \frac{k^2-1}{k^2+1} \ . \tag{105}$$

This factor will appear again in the Doppler effect.*

Page 292

The figure also shows that the time coordinate t_1 assigned by the first observer to the moment in which the light is reflected is different from the coordinate t_2 assigned by the second observer. Time is indeed different for two observers in relative motion. Figure 151 illustrates the result.

The *time dilation factor* between the two time coordinates is found from Figure 150 by comparing the values t_1 and t_2; it is given by

$$\frac{t_1}{t_2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(v) \ . \tag{106}$$

Time intervals for a moving observer are *shorter* by this factor γ; the time dilation factor is always larger than 1. In other words, *moving clocks go slower*. For everyday speeds the

* The explanation of relativity using the factor k is often called *k-calculus*.

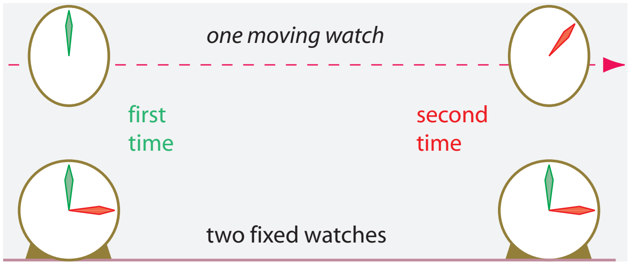

FIGURE 151 Moving clocks go slow

Challenge 580 e effect is tiny. That is why we do not detect time differences in everyday life. Nevertheless, Galilean physics is not correct for speeds near that of light. The same factor γ also appears in the formula $E = \gamma mc^2$, which we will deduce below. Expression (105) or (106) is the only piece of mathematics needed in special relativity: all other results derive from it.

If a light flash is sent forward starting from the second observer and reflected back, he will make the same statement: for him, the first clock is moving, and also for him, the moving clock goes slower. *Each of the observers observes that the other clock goes slower.* The situation is similar to that of two men comparing the number of steps between two identical ladders that are not parallel. A man on either ladder will always observe that the steps of the *other* ladder are shorter. For another analogy, take two people moving away from each other: each of them notes that the other gets smaller as their distance increases.

Naturally, many people have tried to find arguments to avoid the strange conclusion that time differs from observer to observer. But none have succeeded, and experimental results confirm this conclusion. Let us have a look at some of them.

Acceleration of light and the Doppler effect

Light *can* be accelerated. Every mirror does this! We will see in the chapter on electromagnetism that matter also has the power to *bend* light, and thus to accelerate it. However, it Page 580 will turn out that all these methods only change the *direction* of propagation; none has the power to change the *speed* of light in a vacuum. In short, light is an example of a motion Challenge 581 n that cannot be stopped. There are only a few other such examples. Can you name one?

What would happen if we could accelerate light to higher speeds? For this to be possible, light would have to be made of particles with non-vanishing mass. Physicists call such particles *massive* particles. If light had mass, it would be necessary to distinguish the 'massless energy speed' c from the speed of light c_L, which would be lower and would depend on the kinetic energy of those massive particles. The speed of light would not be constant, but the massless energy speed would still be so. Massive light particles could be captured, stopped and stored in a box. Such boxes would make electric illumination unnecessary; it would be sufficient to store some daylight in them and release the light, slowly, during the following night, maybe after giving it a push to speed it up.*

Physicists have tested the possibility of massive light in quite some detail. Observations Ref. 266, Ref. 267 now put any possible mass of light (particles) at less than $1.3 \cdot 10^{-52}$ kg from terrestrial

* Incidentally, massive light would also have *longitudinal* polarization modes. This is in contrast to observations, which show that light is polarized exclusively *transversally* to the propagation direction.

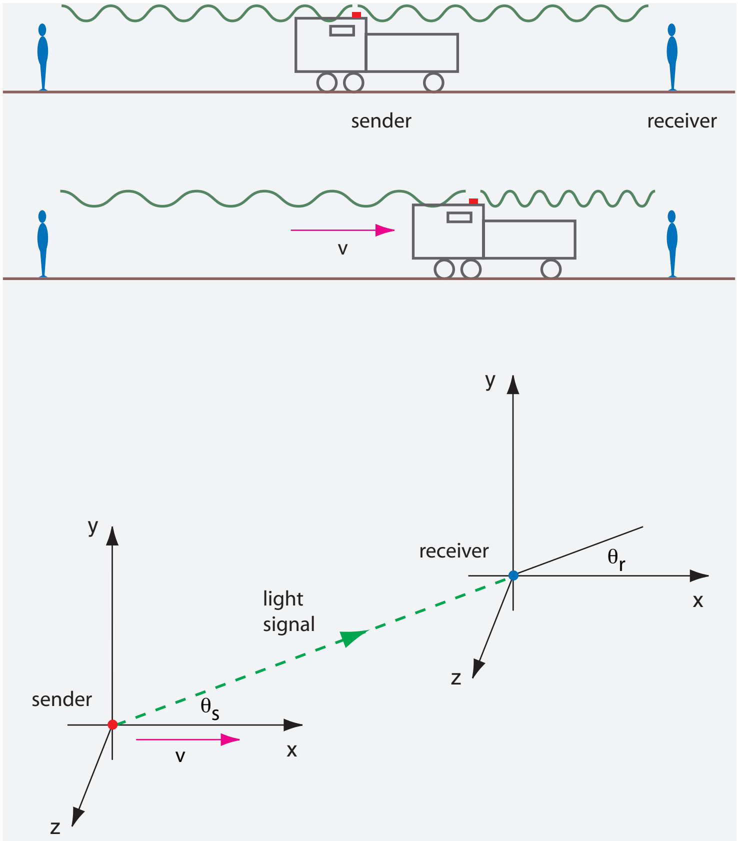

FIGURE 152 The set-up for the observation of the Doppler effect

experiments, and at less than $4 \cdot 10^{-62}$ kg from astrophysical arguments (which are a bit less strict). In other words, light is not heavy, light is light.

But what happens when light hits a *moving* mirror? If the speed of light does not change, something else must. The situation is akin to that of a light source moving with respect to the receiver: the receiver will observe a *different colour* from that observed by the sender. This is called the *Doppler effect*. Christian Doppler* was the first to study the frequency shift in the case of sound waves – the well-known change in whistle tone between approaching and departing trains – and to extend the concept to the case of light waves. As we will see later on, light is (also) a wave, and its colour is determined by its frequency, or equivalently, by its wavelength λ. Like the tone change for moving trains, Doppler realized that a moving light source produces a colour at the receiver that is different from the colour at the source. Simple geometry, and the conservation of the number of maxima and minima, leads to the result

Challenge 582 e

* Christian Andreas Doppler (b. 1803 Salzburg, d. 1853 Venezia), Austrian physicist. Doppler studied the effect named after him for sound and light. In 1842 he predicted (correctly) that one day we would be able to use the effect to measure the motion of distant stars by looking at their colours.

$$\frac{\lambda_r}{\lambda_s} = \frac{1}{\sqrt{1-v^2/c^2}}(1 - \frac{v}{c}\cos\theta_r) = \gamma\,(1 - \frac{v}{c}\cos\theta_r)\ . \tag{107}$$

The variables v and θ_r in this expression are defined in Figure 152. Light from an approaching source is thus blue-shifted, whereas light from a departing source is red-shifted. The first observation of the Doppler effect for light was made by Johannes Stark* in 1905, who studied the light emitted by moving atoms. All subsequent experiments confirmed the calculated colour shift within measurement errors; the latest checks have found agreement to within two parts per million. In contrast to sound waves, a colour change is also found when the motion is *transverse* to the light signal. Thus, a yellow rod in rapid motion across the field of view will have a blue leading edge and a red trailing edge prior to the closest approach to the observer. The colours result from a combination of the longitudinal (first-order) Doppler shift and the transverse (second-order) Doppler shift. At a particular angle $\theta_{unshifted}$ the colours will be the same. (How does the wavelength change in the purely transverse case? What is the expression for $\theta_{unshifted}$ in terms of v?)

Ref. 268

Challenge 583 n

The colour shift is used in many applications. Almost all solid bodies are mirrors for radio waves. Many buildings have doors that open automatically when one approaches. A little sensor above the door detects the approaching person. It usually does this by measuring the Doppler effect of radio waves emitted by the sensor and reflected by the approaching person. (We will see later that radio waves and light are manifestations of the same phenomenon.) So the doors open whenever something moves towards them. Police radar also uses the Doppler effect, this time to measure the speed of cars.**

Page 574

The Doppler effect also makes it possible to measure the velocity of light sources. Indeed, it is commonly used to measure the speed of distant stars. In these cases, the Doppler shift is often characterized by the *red-shift number* z, defined with the help of wavelength λ or frequency F by

$$z = \frac{\Delta\lambda}{\lambda} = \frac{f_S}{f_R} - 1 = \sqrt{\frac{c+v}{c-v}} - 1\ . \tag{108}$$

Can you imagine how the number z is determined? Typical values for z for light sources in the sky range from −0.1 to 3.5, but higher values, up to more than 10, have also been found. Can you determine the corresponding speeds? How can they be so high?

Challenge 585 n

Challenge 586 n

In summary, whenever one tries to change the *speed* of light, one only manages to change its *colour*. That is the Doppler effect.

We know from classical physics that when light passes a large mass, such as a star, it is deflected. Does this deflection lead to a Doppler shift?

Page 135

Challenge 587 n

* Johannes Stark (1874–1957), discovered in 1905 the optical Doppler effect in channel rays, and in 1913 the splitting of spectral lines in electrical fields, nowadays called the Stark effect. For these two discoveries he received the 1919 Nobel Prize for physics. He left his professorship in 1922 and later turned into a full-blown National Socialist. A member of the NSDAP from 1930 onwards, he became known for aggressively criticizing other people's statements about nature purely for ideological reasons; he became rightly despised by the academic community all over the world.

Challenge 584 n

** At what speed does a red traffic light appear green?

The difference between light and sound

The Doppler effect for light is much more important than the Doppler effect for sound. Even if the speed of light were not yet known to be constant, this effect alone would *prove* that time is different for observers moving relative to each other. Why? Time is what we read from our watch. In order to determine whether another watch is synchronized with our own one, we look at both watches. In short, we need to use light signals to synchron-
Ref. 269 ize clocks. Now, any change in the colour of light moving from one observer to another necessarily implies that their watches run differently, and thus that time is *different* for the two of them. One way to see this is to note that also a light source is a clock – 'ticking' very rapidly. So if two observers see different colours from the same source, they measure different numbers of oscillations for the same clock. In other words, time is different for observers moving against each other. Indeed, equation (105) implies that the whole of relativity follows from the full Doppler effect for light. (Can you confirm that the connection between observer-dependent frequencies and observer-dependent time breaks
Challenge 588 n down in the case of the Doppler effect for *sound*?)

Why does the behaviour of light imply special relativity, while that of sound in air does not? The answer is that light is a limit for the motion of energy. Experience shows that there are supersonic aeroplanes, but there are no superluminal rockets. In other words, the limit $v \leqslant c$ is valid only if c is the speed of light, not if c is the speed of sound in air.

However, there is at least one system in nature where the speed of sound is indeed a limit speed for energy: the speed of sound is the limit speed for the motion of *dislocations*
Page 1009 in crystalline solids. (We discuss this in detail later on.) As a result, the theory of special relativity is also valid for such dislocations, provided that the speed of light is replaced everywhere by the speed of sound! Dislocations obey the Lorentz transformations, show
Ref. 270 length contraction, and obey the famous energy formula $E = \gamma m c^2$. In all these effects the speed of sound c plays the same role for dislocations as the speed of light plays for general physical systems.

If special relativity is based on the statement that nothing can move faster than light, this statement needs to be carefully checked.

Can one shoot faster than one's shadow?

> “Quid celerius umbra?* ”

for Lucky Luke to achieve the feat shown in Figure 153, his bullet has to move faster than
Challenge 589 e the speed of light. (What about his hand?) In order to emulate Lucky Luke, we could take the largest practical amount of energy available, taking it directly from an electrical power station, and accelerate the lightest 'bullets' that can be handled, namely electrons. This experiment is carried out daily in particle accelerators such as the Large Electron Positron ring, the LEP, of 27 km circumference, located partly in France and partly in Switzerland, near Geneva. There, 40 MW of electrical power (the same amount used by a small city) accelerates electrons and positrons to energies of over 16 nJ (104.5 GeV) each, and their
Ref. 271 speed is measured. The result is shown in Figure 154: even with these impressive means

* 'What is faster than the shadow?' A motto often found on sundials.

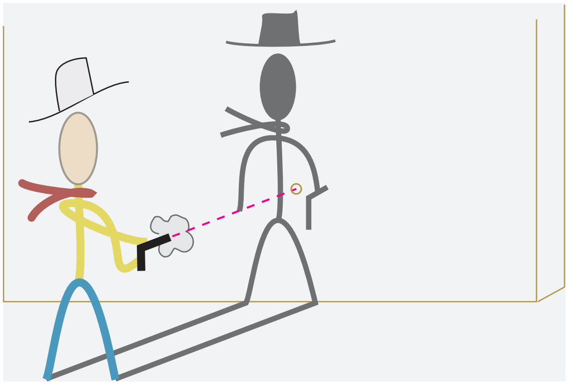

FIGURE 153 Lucky Luke

it is impossible to make electrons move more rapidly than light. (Can you imagine a way
Challenge 590 e to measure energy and speed separately?) The speed–energy relation of Figure 154 is a
Page 321 consequence of the maximum speed, and is deduced below. These and many similar observations thus show that there is a *limit* to the velocity of objects. Bodies (and radiation) cannot move at velocities higher than the speed of light.* The accuracy of Galilean mechanics was taken for granted for more than three centuries, so that nobody ever thought of checking it; but when this was finally done, as in Figure 154, it was found to be wrong.

The people most unhappy with this limit are computer engineers: if the speed limit were higher, it would be possible to make faster microprocessors and thus faster computers; this would allow, for example, more rapid progress towards the construction of computers that understand and use language.

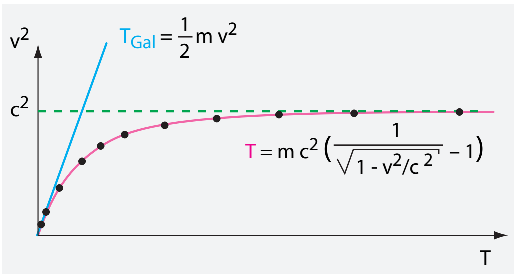

FIGURE 154 Experimental values (dots) for the electron velocity v as function of their kinetic energy T, compared with the prediction of Galilean physics (blue) and that of special relativity (red)

The existence of a limit speed runs counter to Galilean mechanics. In fact, it means that for velocities near that of light, say about 15 000 km/s or more, the expression $mv^2/2$ is *not* equal to the kinetic energy T of the particle. In fact, such high speeds are rather common: many families have an example in their home. Just calculate the speed of electrons inside a television, given that the transformer inside produces
Challenge 591 n 30 kV.

* There are still people who refuse to accept these results, as well as the ensuing theory of relativity. Every physicist should enjoy the experience, at least once in his life, of conversing with one of these men. (Strangely, no woman has yet been reported as belonging to this group of people.) This can be done, for example, via
Ref. 272 the internet, in the sci.physics.relativity newsgroup. See also the http://www.crank.net website. Crackpots are a fascinating lot, especially since they teach the importance of *precision* in language and in reasoning, which they all, without exception, neglect. Encounters with several of them provided the inspiration for this chapter.

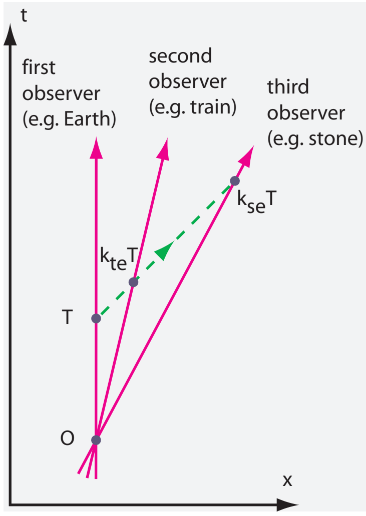

FIGURE 155 How to deduce the composition of velocities

The observation of speed of light as a *limit* speed for objects is easily seen to be a consequence of its *constancy*. Bodies that can be at rest in one frame of reference obviously move more slowly than the maximum velocity (light) in that frame. Now, if something moves more slowly than something else for *one* observer, it does so for all other observers as well. (Trying to imagine a world in which this would not be so is interesting: funny things would happen, such as things interpenetrating each other.) Since the speed of light is the same for all observers, no object can move faster than light, for every observer.

Challenge 592 d

We follow that the maximum speed is the speed of *massless* entities. Electromagnetic waves, including light, are the only known entities that can travel at the maximum speed. Gravitational waves are also predicted to achieve maximum speed. Though the speed of neutrinos cannot be distinguished experimentally from the maximum speed, recent experiments suggest that they do have a tiny mass.

Ref. 273

Conversely, if a phenomenon exists whose speed is the limit speed for one observer, then this limit speed must necessarily be the same for all observers. Is the connection between limit property and observer invariance generally valid in nature?

Challenge 593 e
Challenge 594 r

The composition of velocities

If the speed of light is a limit, no attempt to exceed it can succeed. This implies that when velocities are composed, as when one throws a stone while running, the values cannot simply be added. If a train is travelling at velocity v_{te} relative to the Earth, and somebody throws a stone inside it with velocity v_{st} relative to the train in the same direction, it is usually assumed as evident that the velocity of the stone relative to the Earth is given by $v_{se} = v_{st} + v_{te}$. In fact, both reasoning and measurement show a different result.

The existence of a maximum speed, together with Figure 155, implies that the k-factors must satisfy $k_{se} = k_{st} k_{te}$.* Then we only have to insert the relation (105) between each k-

* By taking the (natural) logarithm of this equation, one can define a quantity, the *rapidity*, that measures

Challenge 595 e factor and the respective speed to get

$$v_{se} = \frac{v_{st} + v_{te}}{1 + v_{st} v_{te}/c^2} . \qquad (109)$$

Challenge 596 e This is called the *velocity composition formula*. The result is never larger than c and is always smaller than the naive sum of the velocities.* Expression (109) has been confirmed by all of the millions of cases for which it has been checked. You may check that it reduces to the naive sum for everyday life values. Page 321, page 544 Ref. 267

Observers and the principle of special relativity

Special relativity is built on a simple principle:

▷ *The maximum speed of energy transport is the same for all observers.*

Ref. 275 Or, as Hendrik Lorentz** liked to say, the equivalent:

▷ *The speed v of a physical system is bound by*

$$v \leqslant c \qquad (110)$$

for all observers, where c is the speed of light.

This independence of the speed of light from the observer was checked with high precision by Michelson and Morley*** in the years from 1887 onwards. It has been confirmed in all subsequent experiments; the most precise to date, which achieved a precision of 10^{-14} is shown in Figure 156. Ref. 276 Ref. 277

In fact, special relativity is also confirmed by all the precision experiments that were performed *before* it was formulated. You can even confirm it yourself at home. The way to do this is shown in the section on electrodynamics. Page 544

The existence of a limit speed has several interesting consequences. To explore them, let us keep the rest of Galilean physics intact.**** The limit speed is the speed of light. It is constant for all observers. This constancy implies:

the speed and is additive.

Ref. 274 * One can also deduce the Lorentz transformation directly from this expression.

** Hendrik Antoon Lorentz (b. 1853 Arnhem, d. 1928 Haarlem) was, together with Boltzmann and Kelvin, one of the most important physicists of his time. He deduced the so-called Lorentz transformation and the Lorentz contraction from Maxwell's equation for the electrodynamic field. He was the first to understand, long before quantum theory confirmed the idea, that Maxwell's equations for the vacuum also describe matter and all its properties, as long as moving charged point particles – the electrons – are included. He showed this in particular for the dispersion of light, for the Zeeman effect, for the Hall effect and for the Faraday effect. He gave the correct description of the Lorentz force. In 1902, he received the physics Nobel Prize, together with Pieter Zeeman. Outside physics, he was active in the internationalization of scientific collaborations. He was also instrumental in the creation of the largest human-made structures on Earth: the polders of the Zuyder Zee.

*** Albert Abraham Michelson (b. 1852 Strelno, d. 1931 Pasadena), Prussian–Polish–US-American physicist, awarded the Nobel Prize in physics in 1907. Michelson called the set-up he devised an *interferometer*, a term still in use today. Edward William Morley (1838–1923), US-American chemist, was Michelson's friend and long-time collaborator.

Page 87 **** This point is essential. For example, Galilean physics states that only *relative* motion is physical. Galilean physics also excludes various mathematically possible ways to realize a constant light speed that would con-

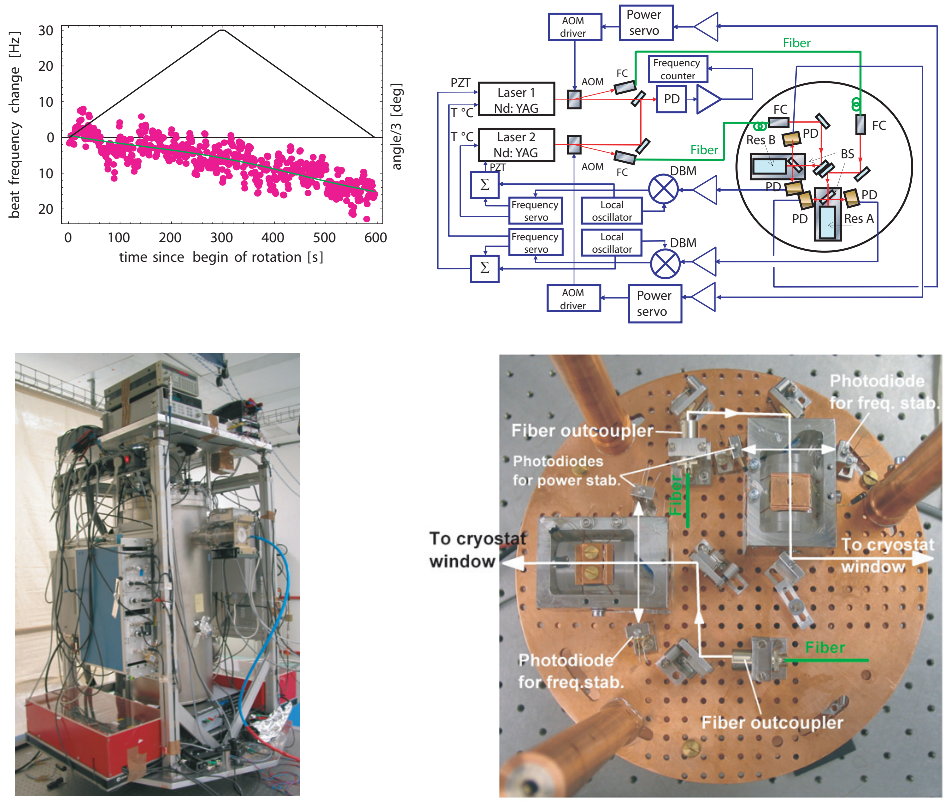

FIGURE 156 The result, the schematics and the cryostat set-up for the most precise Michelson–Morley experiment performed to date (© Stephan Schiller)

— In a closed free-floating room, there is no way to tell the speed of the room.
— There is no notion of absolute rest (or space): rest (like space) is an observer-dependent concept.*
— Time depends on the observer; time is not absolute.

More interesting and specific conclusions can be drawn when two additional conditions are assumed. First, we study situations where gravitation can be neglected. (If this not the case, we need *general* relativity to describe the system.) Secondly, we also assume that the data about the bodies under study – their speed, their position, etc. – can be gathered without disturbing them. (If this not the case, we need *quantum theory* to describe the system.)

To deduce the *precise* way in which the different time intervals and lengths measured by two observers are related to each other, we take an additional simplifying step. We start

tradict everyday life.

Einstein's original 1905 paper starts from two principles: the constancy of the speed of light and the equivalence of all inertial observers. The latter principle had already been stated in 1632 by Galileo; only the constancy of the speed of light was new. Despite this fact, the new theory was named – by Poincaré – after
Ref. 259 the old principle, instead of calling in 'invariance theory', as Einstein would have preferred.
Challenge 597 n * Can you give the argument leading to this deduction?

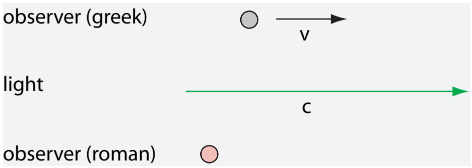

FIGURE 157 Two inertial observers and a beam of light

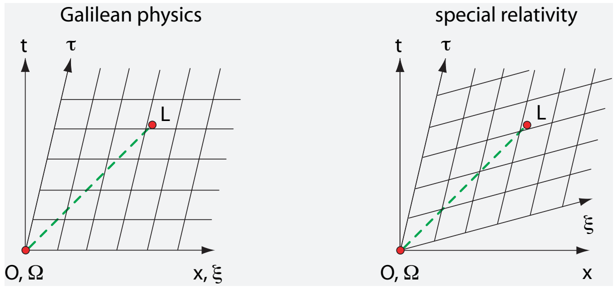

FIGURE 158 Space-time diagrams for light seen from two different observers using coordinates (t, x) and (τ, ξ)

with a situation where no interaction plays a role. In other words, we start with *relativistic kinematics* of bodies moving without disturbance.

If an undisturbed body is observed to travel along a straight line with a constant velocity (or to stay at rest), one calls the observer *inertial*, and the coordinates used by the observer an *inertial frame of reference*. Every inertial observer is itself in undisturbed motion. Examples of inertial observers (or frames) thus include – in *two* dimensions – those moving on a frictionless ice surface or on the floor inside a smoothly running train or ship; for a full example – in all *three* spatial dimensions – we can take a cosmonaut travelling in a space-ship as long as the engine is switched off. Inertial observers in three dimensions might also be called *free-floating* observers. They are thus not so common. Non-inertial observers are much more common. Can you confirm this? Inertial observers are the most simple ones, and they form a special set:

Challenge 598 e

— Any two inertial observers move with constant velocity relative to each other (as longs as gravity plays no role, as assumed above).
— All inertial observers are equivalent: they describe the world with the same equations. Because it implies the lack of absolute space and time, this statement was called the *principle of relativity* by Henri Poincaré. However, the *essence* of relativity is the existence of a limit speed.

To see how measured length and space intervals change from one observer to the other, we assume two inertial observers, a Roman one using coordinates x, y, z and t, and a Greek one using coordinates ξ, υ, ζ and τ,* that move with velocity $\mathbf{v}$ relative to each other. The axes are chosen in such a way that the velocity points in the x-direction. The

* They are read as 'xi', 'upsilon', 'zeta' and 'tau'. The names, correspondences and pronunciations of all Greek letters are explained in Appendix A.

constancy of the speed of light in any direction for any two observers means that for the motion of light the coordinate differentials are related by

$$0 = (c\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2 = (c\mathrm{d}\tau)^2 - (\mathrm{d}\xi)^2 - (\mathrm{d}\upsilon)^2 - (\mathrm{d}\zeta)^2 \ . \qquad (111)$$

Assume also that a flash lamp at rest for the Greek observer, thus with $\mathrm{d}\xi = 0$, produces two flashes separated by a time interval $\mathrm{d}\tau$. For the Roman observer, the flash lamp moves with speed v, so that $\mathrm{d}x = v\mathrm{d}t$. Inserting this into the previous expression, and assuming linearity and speed direction independence for the general case, we find that intervals are

Challenge 599 e

related by

$$\begin{aligned}
\mathrm{d}t &= \gamma(\mathrm{d}\tau + v\mathrm{d}\xi/c^2) = \frac{\mathrm{d}\tau + v\mathrm{d}\xi/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{with} \quad v = \mathrm{d}x/\mathrm{d}t \\
\mathrm{d}x &= \gamma(\mathrm{d}\xi + v\mathrm{d}\tau) = \frac{\mathrm{d}\xi + v\mathrm{d}\tau}{\sqrt{1 - v^2/c^2}} \\
\mathrm{d}y &= \mathrm{d}\upsilon \\
\mathrm{d}z &= \mathrm{d}\zeta \ .
\end{aligned} \qquad (112)$$

These expressions describe how length and time intervals measured by different observers are related. At relative speeds v that are small compared to the velocity of light, such as occur in everyday life, the time intervals are essentially equal; the *stretch factor* or *relativistic correction* or *relativistic contraction* γ is then equal to 1 for all practical purposes. However, for velocities *near* that of light the measurements of the two observers give different values. In these cases, space and time *mix*, as shown in Figure 158.

The expressions (112) are also strange in another respect. When two observers look

Challenge 600 n

at each other, each of them claims to measure shorter intervals than the other. In other words, special relativity shows that the grass on the other side of the fence is always *shorter* – if one rides along beside the fence on a bicycle and if the grass is inclined. We explore this bizarre result in more detail shortly.

The stretch factor γ is equal to 1 for most practical purposes in everyday life. The largest value humans have ever achieved is about $2 \cdot 10^5$; the largest observed value in nature is

Challenge 601 n

about 10^{12}. Can you imagine where they occur?

Once we know how space and time *intervals* change, we can easily deduce how *coordinates* change. Figures 157 and 158 show that the x coordinate of an event L is the sum of two intervals: the ξ coordinate and the length of the distance between the two origins. In other words, we have

$$\xi = \gamma(x - vt) \quad \text{and} \quad v = \frac{\mathrm{d}x}{\mathrm{d}t} \ . \qquad (113)$$

Using the invariance of the space-time interval, we get

$$\tau = \gamma(t - xv/c^2) \ . \qquad (114)$$

Henri Poincaré called these two relations the *Lorentz transformations of space and time* after their discoverer, the Dutch physicist Hendrik Antoon Lorentz.* In one of the most beautiful discoveries of physics, in 1892 and 1904, Lorentz deduced these relations from the equations of electrodynamics, where they had been lying, waiting to be discovered, since 1865.** In that year James Clerk Maxwell had published the equations in order to describe everything electric and magnetic. However, it was Einstein who first understood that t and τ, as well as x and ξ, are equally correct and thus equally valid descriptions of space and time.

Ref. 278
Page 555

The Lorentz transformation describes the change of viewpoint from one inertial frame to a second, moving one. This change of viewpoint is called a (Lorentz) *boost*. The formulae (113) and (114) for the boost are central to the theories of relativity, both special and general. In fact, the mathematics of special relativity will not get more difficult than that: if you know what a square root is, you can study special relativity in all its beauty.

Many alternative formulae for boosts have been explored, such as expressions in which the relative acceleration of the two observers is included, as well as the relative velocity. However, they had all to be discarded after comparing their predictions with experimental results. Before we have a look at such experiments, we continue with a few logical deductions from the boost relations.

Ref. 279

What is space-time?

> “Von Stund’ an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbstständigkeit bewaren.***
>
> Hermann Minkowski.”

The Lorentz transformations tell us something important: that space and time are two aspects of the same basic entity. They ‘mix’ in different ways for different observers. This fact is commonly expressed by stating that time is the *fourth dimension*. This makes sense because the common basic entity – called *space-time* – can be defined as the set of all events, events being described by four coordinates in time and space, and because the set of all events has the properties of a manifold.**** (Can you confirm this?)

Challenge 602 n

In other words, the existence of a maximum speed in nature forces us to introduce a space-time manifold for the description of nature. In the theory of special relativity, the space-time manifold is characterized by a simple property: the *space-time interval* $\mathrm{d}i$ between two nearby events, defined as

Ref. 280

$$\mathrm{d}i^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = c^2\mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right) , \tag{115}$$

* For information about Hendrik Antoon Lorentz, see page 298.

** The same discovery had been published first in 1887 by the German physicist Woldemar Voigt (1850–1919); Voigt – pronounced ‘Fohgt’ – was also the discoverer of the Voigt effect and the Voigt tensor. Independently, in 1889, the Irishman George F. Fitzgerald also found the result.

*** ‘Henceforth space by itself and time by itself shall completely fade into shadows and only a kind of union of the two shall preserve autonomy.’ This famous statement was the starting sentence of Minkowski’s 1908 talk at the meeting of the Gesellschaft für Naturforscher und Ärzte.

**** The term ‘manifold’ is defined in Appendix D.

Page 1231

is independent of the (inertial) observer. Such a space-time is also called Minkowski space-time, after Hermann Minkowski* the teacher of Albert Einstein; he was the first, in 1904, to define the concept of space-time and to understand its usefulness and importance.

The space-time interval $\mathrm{d}i$ of equation (115) has a simple interpretation. It is the time measured by an observer moving from event (t, x) to event $(t+\mathrm{d}t, x+\mathrm{d}x)$, the so-called *proper time*, multiplied by c. If we neglect the factor c, we could simply call it wristwatch time.

We *live* in a Minkowski space-time, so to speak. Minkowski space-time exists independently of things. And even though coordinate systems can be different from observer to observer, the underlying entity, space-time, is still *unique*, even though space and time by themselves are not.

How does Minkowski space-time differ from Galilean space-time, the combination of everyday space and time? Both space-times are manifolds, i.e. continuum sets of points, both have one temporal and three spatial dimensions, and both manifolds have the topology of the punctured sphere. (Can you confirm this?) Both manifolds are flat, i.e. free of curvature. In both cases, space is what is measured with a metre rule or with a light ray, and time is what is read from a clock. In both cases, space-time is fundamental; it is and remains the *background* and the *container* of things and events.

Challenge 603 n

The central difference, in fact the only one, is that Minkowski space-time, in contrast to the Galilean case, *mixes* space and time, and in particular, does so differently for observers with different speeds, as shown in Figure 158. That is why time is an observer-dependent concept.

The maximum speed in nature thus forces us to describe motion with space-time. That is interesting, because in space-time, speaking in simple terms, *motion does not exist*. Motion exists only in space. In space-time, nothing moves. For each point particle, space-time contains a world-line. In other words, instead of asking why motion exists, we can equivalently ask why space-time is criss-crossed by world-lines. At this point, we are still far from answering either question. What we can do is to explore *how* motion takes place.

Can we travel to the past? – Time and causality

We know that time is different for different observers. Does time nevertheless order events in sequences? The answer given by relativity is a clear 'yes and no'. Certain sets of events are not naturally ordered by time; others sets are. This is best seen in a space-time diagram.

Clearly, two events can be placed in sequence only if one event is the *cause* of the other. But this connection can only apply if the events exchange energy (e.g. through a signal). In other words, a relation of cause and effect between two events implies that energy or signals can travel from one event to the other; therefore, the speed connecting the two events must not be larger than the speed of light. Figure 159 shows that event E at the origin of the coordinate system can only be influenced by events in quadrant IV (the *past light cone*, when all space dimensions are included), and can itself influence only events

* Hermann Minkowski (1864–1909), German mathematician. He had developed similar ideas to Einstein, but the latter was faster. Minkowski then developed the concept of space-time. Minkowski died suddenly at the age of 44.

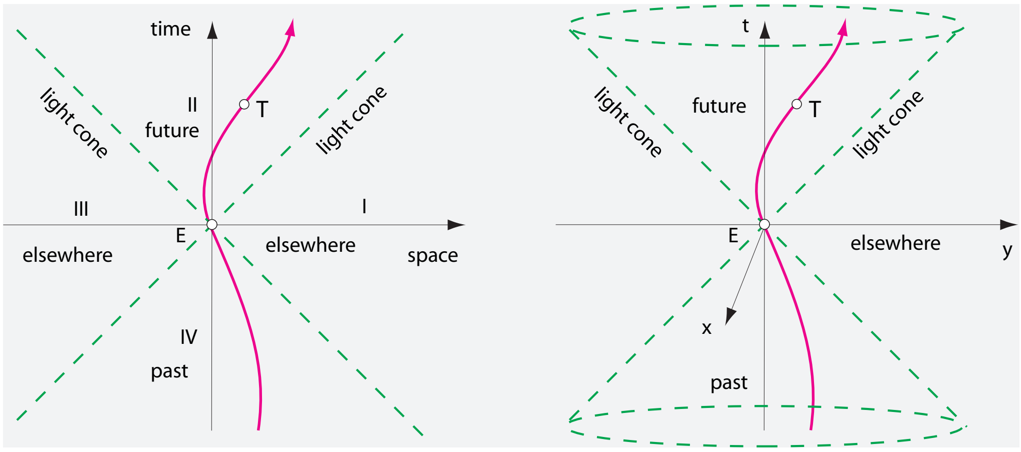

FIGURE 159 A space-time diagram for a moving object T seen from an inertial observer O in the case of one and two spatial dimensions

in quadrant II (the *future light cone*). Events in quadrants I and III neither influence nor are influenced by event E. The light cone defines the boundary between events that *can* be ordered with respect to their origin – namely those inside the cone – and those that *cannot* – those outside the cones, happening elsewhere for all observers. (Some people call all the events happening elsewhere the *present*.) So, time orders events only *partially*. For example, for two events that are not causally connected, their temporal order (or their simultaneity) depends on the observer!

In particular, the past light cone gives the complete set of events that can influence what happens at the origin. One says that the origin is *causally connected* only to the past light cone. This statement reflects the fact that any influence involves transport of energy, and thus cannot travel faster than the speed of light. Note that causal connection is an invariant concept: all observers agree on whether or not it applies to two given events. Can you confirm this? Challenge 604 n

A vector inside the light cone is called *timelike*; one on the light cone is called *lightlike* or *null*; and one outside the cone is called *spacelike*. For example, the *world-line* of an observer, i.e. the set of all events that make up its past and future history, consists of time-like events only. Time is the fourth dimension; it expands space to space-time and thus 'completes' space-time. This is the relevance of the fourth dimension to special relativity, no more and no less.

Special relativity thus teaches us that causality and time can be defined *only* because light cones exist. If transport of energy at speeds faster than that of light did exist, time could not be defined. Causality, i.e. the possibility of (partially) ordering events for all observers, is due to the existence of a maximal speed.

If the speed of light could be surpassed in some way, the future could influence the past. Can you confirm this? In such situations, one would observe *acausal* effects. However, there is an everyday phenomenon which tells that the speed of light is indeed maximal: our memory. If the future could influence the past, we would also be able to *remember* the future. To put it in another way, if the future could influence the past, the second principle Challenge 605 n of thermodynamics would not be valid and our memory would not work.* No other data from everyday life or from experiments provide any evidence that the future can influence the past. In other words, *time travel to the past is impossible*. How the situation changes in quantum theory will be revealed later on. Interestingly, time travel to the future *is* possible, as we will see shortly.

Curiosities of special relativity

Faster than light: how far can we travel?

How far away from Earth can we travel, given that the trip should not last more than a lifetime, say 80 years, and given that we are allowed to use a rocket whose speed can approach the speed of light as closely as desired? Given the time t we are prepared to spend in a rocket, given the speed v of the rocket and assuming optimistically that it can accelerate and decelerate in a negligible amount of time, the distance d we can move away

Challenge 606 e

is given by

$$d = \frac{vt}{\sqrt{1 - v^2/c^2}} . \tag{116}$$

The distance d is larger than ct already for $v > 0.71c$, and, if v is chosen large enough, it increases beyond all bounds! In other words, relativity does *not* limit the distance we can travel in a lifetime, and not even the distance we can travel in a single second. We could, in principle, roam the entire universe in less than a second. In situations such as these it makes sense to introduce the concept of *proper velocity* w, defined as

$$w = d/t = \frac{v}{\sqrt{1 - v^2/c^2}} = \gamma\, v . \tag{117}$$

As we have just seen, proper velocity is *not* limited by the speed of light; in fact the proper velocity of light itself is infinite.**

Synchronization and time travel – can a mother stay younger than her own daughter?

A maximum speed implies that time is different for different observers moving relative to each other. So we have to be careful about how we synchronize clocks that are far apart, even if they are at rest with respect to each other in an inertial reference frame. For example, if we have two similar watches showing the same time, and if we carry one of them for a walk and back, they will show different times afterwards. This experiment has

* Another related result is slowly becoming common knowledge. Even if space-time had a nontrivial shape, such as a cylindrical topology with closed time-like curves, one still would not be able to travel into the past, in contrast to what many science fiction novels suggest. This is made clear by Stephen Blau in a recent pedagogical paper.

Ref. 281

** Using proper velocity, the relation given in equation (109) for the superposition of two velocities $\mathbf{w}_a = \gamma_a \mathbf{v}_a$ and $\mathbf{w}_b = \gamma_b \mathbf{v}_b$ simplifies to

Challenge 607 e

$$w_{s\parallel} = \gamma_a \gamma_b (v_a + v_{b\parallel}) \quad \text{and} \quad w_{s\perp} = w_{b\perp} , \tag{118}$$

where the signs $\parallel$ and $\perp$ designate the component in the direction of and the component perpendicular to $\mathbf{v}_a$, respectively. One can in fact express all of special relativity in terms of 'proper' quantities.

Ref. 282

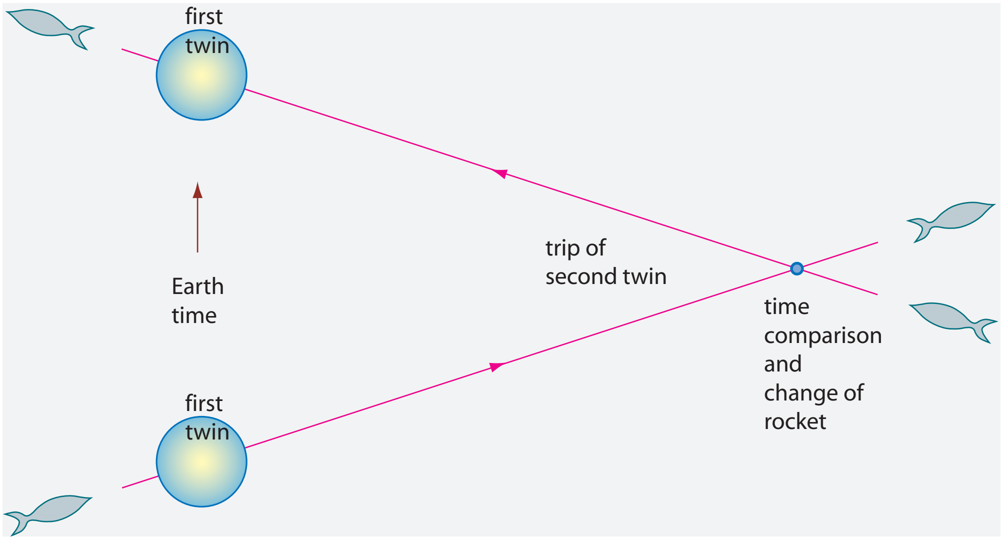

FIGURE 160 The twin paradox

Ref. 283, Ref. 284 actually been performed several times and has fully confirmed the prediction of special relativity. The time difference for a person or a watch in an aeroplane travelling around the Earth once, at about 900 km/h, is of the order of 100 ns – not very noticeable in everyday life. In fact, the delay is easily calculated from the expression

$$\frac{t}{t'} = \gamma \; . \qquad (119)$$

Human bodies are clocks; they show the elapsed time, usually called *age*, by various changes in their shape, weight, hair colour, etc. If a person goes on a long and fast trip, on her return she will have aged *less* than a second person who stayed at her (inertial) home.

The most famous illustration of this is the famous *twin paradox* (or *clock paradox*). An adventurous twin jumps on a relativistic rocket that leaves Earth and travels for many years. Far from Earth, he jumps on another relativistic rocket going the other way and returns to Earth. The trip is illustrated in Figure 160. At his arrival, he notes that his twin brother on Earth is much older than himself. Can you explain this result, especially the asymmetry between the two brothers? This result has also been confirmed in many
Ref. 285 experiments.

Special relativity thus confirms, in a surprising fashion, the well-known observation that those who travel a lot remain younger. The price of the retained youth is, however, that everything around one changes very much more quickly than if one is at rest with the environment.

The twin paradox can also be seen as a confirmation of the possibility of time travel to the future. With the help of a fast rocket that comes back to its starting point, we can arrive at local times that we would never have reached within our lifetime by staying home. Alas, we can *never* return to the past.*

Ref. 286 * There are even special books on time travel, such as the well researched text by Nahin. Note that the concept

One of the simplest experiments confirming the prolonged youth of fast travellers involves the counting of muons. Muons are particles that are continuously formed in the upper atmosphere by cosmic radiation. Page 919 Muons *at rest* (with respect to the measuring clock) have a finite half-life of 2.2 µs (or, at the speed of light, 660 m). After this amount of time, half of the muons have decayed. This half-life can be measured using simple muon counters. In addition, there exist special counters that only count muons travelling within a certain speed range, say from $0.9950c$ to $0.9954c$. One can put one of these special counters on top of a mountain and put another in the valley below, as shown in Figure 161. The first time this experiment Ref. 287 was performed, the height difference was 1.9 km. Flying 1.9 km through the atmosphere at the mentioned speed takes about 6.4 µs. With the half-life just given, a naive calculation finds that only about Challenge 608 n 13% of the muons observed at the top should arrive at the lower site. However, it is observed that about 82% of the muons arrive below. The reason for this result is the relativistic time dilation. Indeed, at the mentioned speed, muons experience a time difference of only 0.62 µs during the travel from the mountain top to the valley. This shorter time yields a much lower number of lost muons than would be the case without time dilation; moreover, the measured percentage confirms the value of the predicted time dilation Challenge 609 n factor γ within experimental errors, as you may want to check. A similar effect is seen when relativistic muons are produced in accelerators.

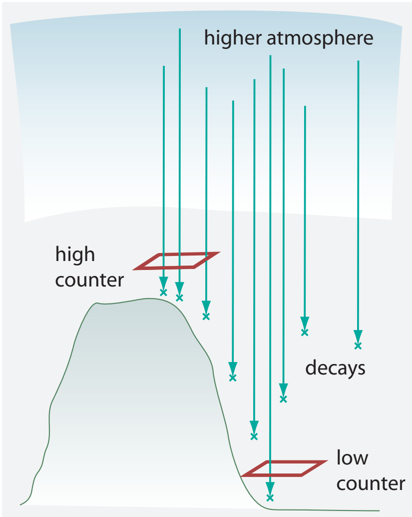

FIGURE 161 More muons than expected arrive at the ground because fast travel keeps them young

Half-life dilation has also been found for many other decaying systems, such as pions, hydrogen atoms, neon atoms and various nuclei, always confirming the predictions of special relativity. Since all bodies in nature are made of particles, the 'youth effect' of high speeds (usually called 'time dilation') applies to bodies of all sizes; indeed, it has not only Ref. 267 been observed for particles, but also for lasers, radio transmitters and clocks.

If motion leads to time dilation, a clock on the Equator, constantly running around the Earth, should go slower than one at the poles. However, this prediction, which was Ref. 288 made by Einstein himself, is incorrect. The centrifugal acceleration leads to a reduction in gravitational acceleration that exactly cancels the increase due to the velocity. This story serves as a reminder to be careful when applying special relativity in situations involving gravity. Special relativity is only applicable when space-time is flat, not when gravity is present.

In short, a mother *can* stay younger than her daughter. We can also conclude that we cannot synchronize clocks at rest with respect to each other simply by walking, clock in hand, from one place to another. The correct way to do so is to exchange light signals. Challenge 610 n Can you describe how?

of time travel has to be clearly defined; otherwise one has no answer to the clerk who calls his office chair a time machine, as sitting on it allows him to get to the future.

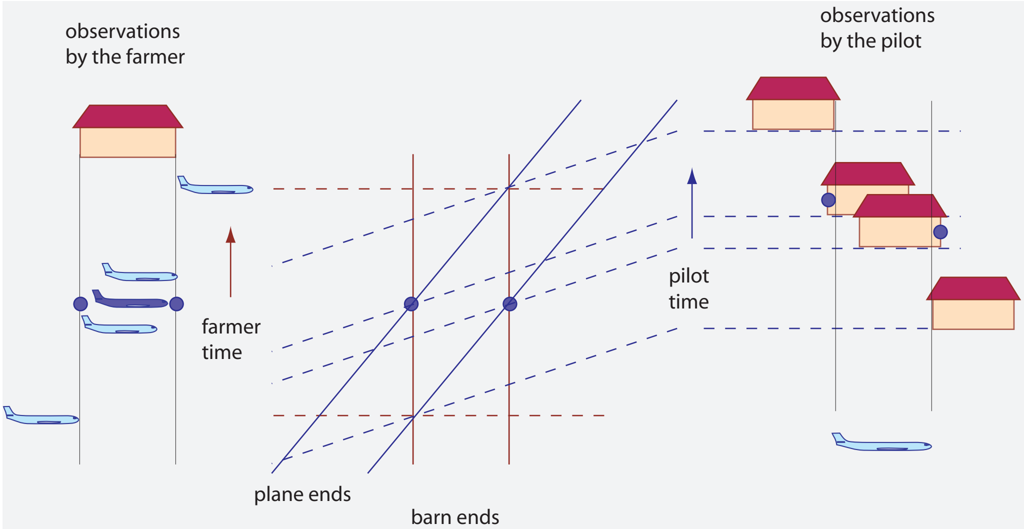

FIGURE 162 The observations of the pilot and the barn owner

A precise definition of synchronization allows us to call two distant events simultaneous. In addition, special relativity shows that simultaneity depends on the observer. This is confirmed by all experiments performed so far.

However, the mother's wish is not easy to fulfil. Let us imagine that a mother is accelerated in a spaceship away from Earth at $10\,\mathrm{m/s^2}$ for ten years, then decelerates at $10\,\mathrm{m/s^2}$ for another ten years, then accelerates for ten additional years towards the Earth, and finally decelerates for ten final years in order to land safely back on our planet. The mother has taken 40 years for the trip. She got as far as 22 000 light years from Earth. At her return on Earth, 44 000 years have passed. All this seems fine, until we realize that the necessary amount of fuel, even for the most efficient engine imaginable, is so large that the mass returning from the trip is only one part in $2 \cdot 10^{19}$. The necessary amount of fuel does not exist on Earth.

Challenge 611 e

Length contraction

The length of an object measured by an observer attached to the object is called its proper length. According to special relativity, the length measured by an inertial observer passing by is always smaller than the proper length. This result follows directly from the Lorentz transformations.

Challenge 612 e

For a Ferrari driving at 300 km/h or 83 m/s, the length is contracted by 0.15 pm: less than the diameter of a proton. Seen from the Sun, the Earth moves at 30 km/s; this gives a length contraction of 6 cm. Neither of these effects has ever been measured. But larger effects could be. Let us explore some examples.

Imagine a pilot flying through a barn with two doors, one at each end. The plane is slightly longer than the barn, but moves so rapidly that its relativistically contracted length is shorter than the length of the barn. Can the farmer close the barn (at least for a short time) with the plane completely inside? The answer is positive. But why can the pilot not say the following: relative to him, the barn is contracted; therefore the plane

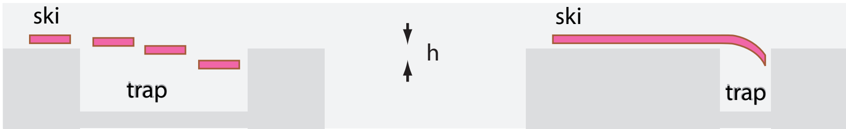

FIGURE 163 The observations of the trap digger and of the snowboarder, as (misleadingly) published in the literature

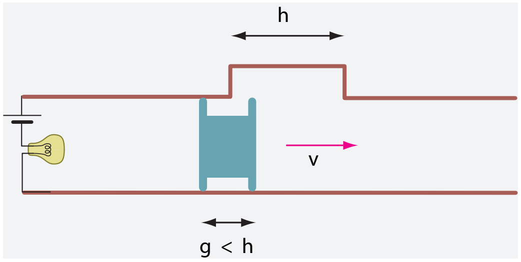

FIGURE 164 Does the conducting glider keep the lamp lit at large speeds?

FIGURE 165 What happens to the rope?

does not fit inside the barn? The answer is shown in Figure 162. For the farmer, the doors close (and reopen) at the same time. For the pilot, they do not. For the farmer, the pilot is in the dark for a short time; for the pilot, the barn is never dark. (That is not completely Challenge 613 n true: can you work out the details?)

We now explore some variations of the general case. Can a rapid snowboarder fall into a hole that is a bit shorter than his board? Imagine him boarding so fast that the length contraction factor $\gamma = d/d'$ is 4.* For an observer on the ground, the snowboard is four times shorter, and when it passes over the hole, it will fall into it. However, for the boarder, it is the hole which is four times shorter; it seems that the snowboard cannot fall into it.

Ref. 289 More careful analysis shows that, in contrast to the observation of the hole digger, the snowboarder does not experience the board's shape as fixed: while passing over the hole, the boarder observes that the board takes on a parabolic shape and falls into the hole, Challenge 615 e as shown in Figure 163. Can you confirm this? In other words, shape is not an observer-invariant concept. (However, rigidity *is* observer-invariant, if defined properly; can you Challenge 616 n confirm this?)

This explanation, though published, is not correct, as Harald van Lintel and Christian Ref. 290 Gruber have pointed out. One should not forget to estimate the size of the effect. At relativistic speeds the time required for the hole to affect the full thickness of the board cannot be neglected. The snowboarder only sees his board take on a parabolic shape if it is extremely thin and flexible. For usual boards moving at relativistic speeds, the snow-Challenge 617 ny boarder has no time to fall any appreciable height h or to bend into the hole before passing it. Figure 163 is so exaggerated that it is incorrect. The snowboarder would simply speed over the hole.

The paradoxes around length contraction become even more interesting in the case of Ref. 291 a conductive glider that makes electrical contact between two rails, as shown in Figure 164. The two rails are parallel, but one rail has a gap that is longer than the glider. Can you

Challenge 614 n * Even the Earth contracts in its direction of motion around the Sun. Is the value measurable?

work out whether a lamp connected in series stays lit when the glider moves along the
Challenge 618 n rails with relativistic speed? (Make the simplifying and not fully realistic assumption that electrical current flows as long and as soon as the glider touches the rails.) Do you get the same result for all observers? And what happens when the glider is longer than the detour? (Warning: this problem gives rise to *heated* debates!) What is unrealistic in this experiment?

Ref. 292 Another example of length contraction appears when two objects, say two cars, are connected over a distance d by a straight rope, as shown in Figure 165 Imagine that both are at rest at time $t = 0$ and are accelerated together in exactly the same way. The observer at rest will maintain that the two cars remain the same distance apart. On the other hand, the rope needs to span a distance $d' = d/\sqrt{1 - v^2/c^2}$, and thus has to expand when the two cars are accelerating. In other words, the rope will break. Is this prediction confirmed
Challenge 619 n by observers on each of the two cars?

A funny – but quite unrealistic – example of length contraction is that of a submarine
Ref. 293 moving horizontally. Imagine that the resting submarine has tuned its weight to float in water without any tendency to sink or to rise. Now the submarine moves (possibly with relativistic speed). The captain observes the water outside to be Lorentz contracted; thus the water is denser and he concludes that the submarine will rise. A nearby fish sees the submarine to be contracted, thus denser than water, and concludes that the submarine
Challenge 620 n will sink. Who is wrong, and what is the buoyancy force? Alternatively, answer the fol-
Challenge 621 n lowing question: why is it impossible for a submarine to move at relativistic speed?

In summary, length contraction can almost never be realistically observed for macroscopic bodies. However, it does play an important role for *images*.

Relativistic films – aberration and Doppler effect

We have encountered several ways in which observations change when an observer moves at high speed. First of all, Lorentz contraction and aberration lead to *distorted* images. Secondly, aberration increases the viewing angle beyond the roughly 180 degrees that humans are used to in everyday life. A relativistic observer who looks in the direction of motion sees light that is invisible for a resting observer, because for the latter, it comes from behind. Thirdly, the Doppler effect produces *colour-shifted* images. Fourthly, the rapid motion changes the *brightness* and *contrast* of the image: the so-called *searchlight effect*. Each of these changes depends on the direction of sight; they are shown in Figure 167.

Modern computers enable us to simulate the observations made by rapid observers with photographic quality, and even to produce simulated films.* The images of Figure 166 are particularly helpful in allowing us to understand image distortion. They show the viewing angle, the circle which distinguish objects in front of the observer from those behind the observer, the coordinates of the observer's feet and the point on the horizon toward which the observer is moving. Adding these markers in your head when watching other pictures or films may help you to understand more clearly what they show.

* See for example images and films at http://www.anu.edu.au/Physics/Searle by Anthony Searle, at http://www.tat.physik.uni-tuebingen.de/~weiskopf/gallery/index.html by Daniel Weiskopf, at http://www.itp.uni-hannover.de/~dragon/stonehenge/stone1.htm by Norbert Dragon and Nicolai Mokros, or at http://www.tempolimit-lichtgeschwindigkeit.de by Hanns Ruder's group.

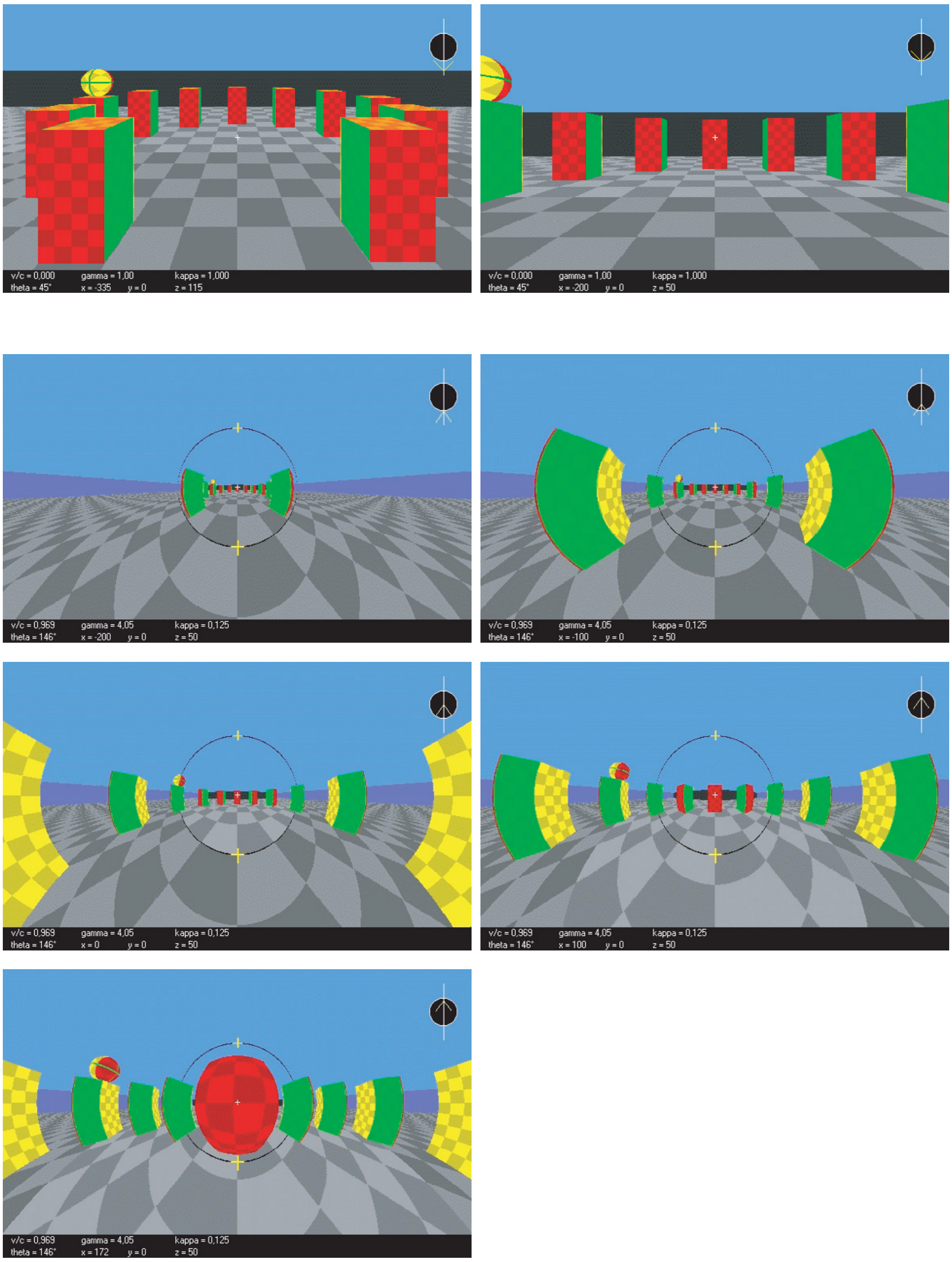

FIGURE 166 Flying through twelve vertical columns (shown in the two uppermost images) with 0.9 times the speed of light as visualized by Nicolai Mokros and Norbert Dragon, showing the effect of speed and position on distortions (© Nicolai Mokros)

FIGURE 167 Flying through three straight and vertical columns with 0.9 times the speed of light as visualized by Daniel Weiskopf: on the left with the original colours; in the middle including the Doppler effect; and on the right including brightness effects, thus showing what an observer would actually see (© Daniel Weiskopf)

FIGURE 168 What a researcher standing and one running rapidly through a corridor observe (ignoring colour effects) (© Daniel Weiskopf)

We note that the shape of the image seen by a moving observer is a *distorted* version of that seen by one at rest at the same point. A moving observer, however, does not see different things than a resting one at the same point. Indeed, light cones are independent of observer motion.

The Lorentz contraction is measurable; however, it cannot be photographed. This surprising distinction was discovered only in 1959. Measuring implies simultaneity at the object's position; photographing implies simultaneity at the observer's position. On a photograph, the Lorentz contraction is modified by the effects due to different light travel times

Ref. 294

from the different parts of an object; the result is a change in shape that is reminiscent of, but not exactly the same as, a rotation. The total deformation is an angle-dependent aberration. We discussed aberration at the beginning of this section. Aberration transforms circles into circles: such transformations are called *conformal*.

Page 285

The images of Figure 168, produced by Daniel Weiskopf, also include the Doppler effect and the brightness changes. They show that these effects are at least as striking as the distortion due to aberration.

This leads to the 'pearl necklace paradox'. If the relativistic motion transforms spheres into spheres, and rods into shorter rods, what happens to a pearl necklace moving along its own long axis? Does it get shorter?

Challenge 622 n

There is much more to be explored using relativistic films. For example, the author predicts that films of rapidly rotating spheres in motion will reveal interesting effects. Also in this case, optical observation and measurement results will differ. For certain combinations of relativistic rotations and relativistic boosts, it is predicted* that the sense of rotation (clockwise or anticlockwise) will *differ* for different observers. This effect will play an interesting role in the discussion of unification.

Challenge 623 r

Which is the best seat in a bus?

Ref. 292

Let us explore another surprise of special relativity. Imagine two twins inside two identically accelerated cars, one in front of the other, starting from standstill at time $t = 0$, as described by an observer at rest with respect to both of them. (There is no connecting rope now.) Both cars contain the same amount of fuel. We easily deduce that the acceleration of the two twins stops, when the fuel runs out, at the same time in the frame of the outside observer. In addition, the distance between the cars has remained the same all along for the outside observer, and the two cars continue rolling with an identical constant velocity v, as long as friction is negligible. If we call the events at which the front car and back car engines switch off f and b, their time coordinates in the outside frame are related simply by $t_f = t_b$. By using the Lorentz transformations you can deduce for the frame of the freely rolling twins the relation

Challenge 624 e

Challenge 625 e

$$t_b = \gamma \Delta x\, v / c^2 + t_f \, , \tag{120}$$

which means that the front twin has aged *more* than the back twin! Thus, in accelerated systems, ageing is position-dependent.

For choosing a seat in a bus, though, this result does not help. It is true that the best seat in an accelerating bus is the back one, but in a decelerating bus it is the front one. At the end of a trip, the choice of seat does not matter.

Is it correct to deduce that people on high mountains age faster than people in valleys, so that living in a valley helps postponing grey hair?

Challenge 626 n

How fast can one walk?

To walk means to move the feet in such a way that at least one of them is on the ground at any time. This is one of the rules athletes have to follow in Olympic walking competitions;

* In July 2005.

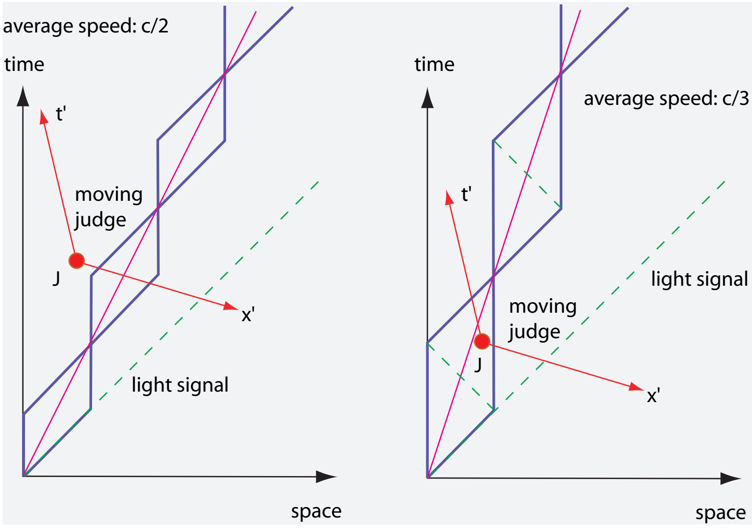

FIGURE 169 For the athlete on the left, the judge moving in the opposite direction sees both feet off the ground at certain times, but not for the athlete on the right

they are disqualified if they break it. A student athlete was thinking about the theoretical maximum speed he could achieve in the Olympics. The ideal would be that each foot accelerates instantly to (almost) the speed of light. The highest walking speed is achieved by taking the second foot off the ground at exactly the same instant at which the first is put down. By 'same instant', the student originally meant 'as seen by a competition judge at rest with respect to Earth'. The motion of the feet is shown in the left diagram of Figure 169; it gives a limit speed for walking of half the speed of light. But then the student noticed that a *moving* judge will see both feet off the ground and thus disqualify the athlete for running. To avoid disqualification by *any* judge, the second foot has to wait for a light signal from the first. The limit speed for Olympic walking is thus only one third of the speed of light.

Ref. 295

Is the speed of shadow greater than the speed of light?

Actually, motion faster than light does exist and is even rather common. Special relativity only constrains the motion of mass and energy. However, non-material points or non-energy-transporting features and images *can* move faster than light. There are several simple examples. To be clear, we are not talking about *proper* velocity, which in these cases cannot be defined anyway. (Why?)

Page 305
Challenge 627 n

The following examples show speeds that are genuinely higher than the speed of light in vacuum.

Consider the point marked X in Figure 170, the point at which scissors cut paper. If the scissors are closed rapidly enough, the point moves faster than light. Similar examples can also be found in every window frame, and in fact in any device that has twisting parts.

Another example of superluminal motion is a music record – an old-fashioned LP – disappearing into its sleeve, as shown in Figure 171. The point where the edge of the record

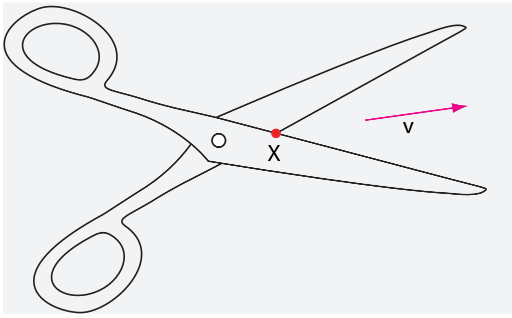

FIGURE 170 A simple example of motion that is faster than light

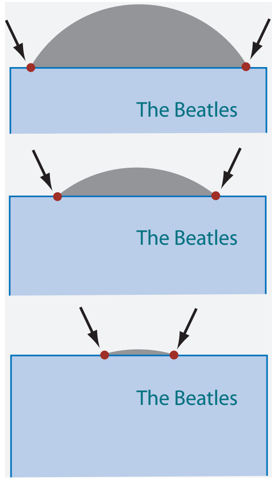

FIGURE 171 Another example of faster-than-light motion

meets the edge of the sleeve can travel faster than light.

Another example suggests itself when we remember that we live on a spherical planet. Imagine you lie on the floor and stand up. Can you show that the initial speed with which the horizon moves away from you can be larger than that of light? Challenge 628 n

Finally, a standard example is the motion of a spot of light produced by shining a laser beam onto the Moon. If the laser is moved, the spot can easily move faster than light. The same applies to the light spot on the screen of an oscilloscope when a signal of sufficiently high frequency is fed to the input.

All these are typical examples of the *speed of shadows*, sometimes also called the *speed of darkness*. Both shadows and darkness can indeed move faster than light. In fact, there is no limit to their speed. Can you find another example? Challenge 629 n

In addition, there is an ever-increasing number of experimental set-ups in which the phase velocity or even the group velocity of light is higher than c. They regularly make headlines in the newspapers, usually along the lines of 'light moves faster than light'. We will discuss this surprising phenomenon in more detail later on. Page 591 In fact, these cases can also be seen – with some imagination – as special cases of the 'speed of shadow' phenomenon.

For a different example, imagine we are standing at the exit of a tunnel of length l. We see a car, whose speed we know to be v, entering the other end of the tunnel and driving towards us. We know that it entered the tunnel because the car is no longer in the Sun or because its headlights were switched on at that moment. At what time t, after we see it entering the tunnel, does it drive past us? Simple reasoning shows that t is given by

$$t = l/v - l/c \,. \tag{121}$$

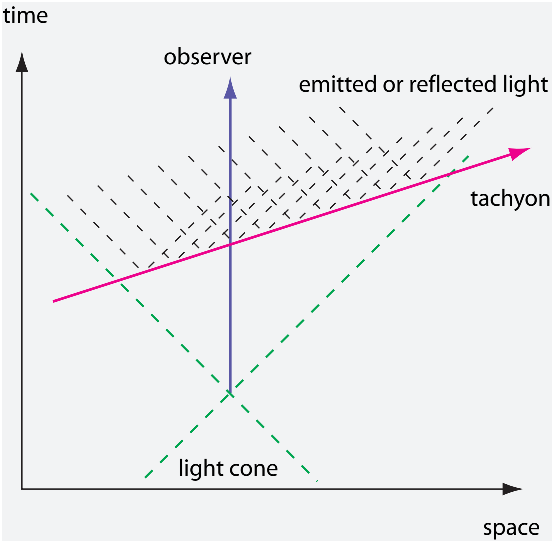

FIGURE 172 Hypothetical space-time diagram for tachyon observation

In other words, the approaching car seems to have a velocity v_{appr} of

$$v_{appr} = \frac{l}{t} = \frac{vc}{c - v} \,, \tag{122}$$

which is higher than c for any car velocity v higher than $c/2$. For cars this does not happen too often, but astronomers know a type of bright object in the sky called a *quasar* (a contraction of 'quasi-stellar object'), which sometimes emits high-speed gas jets. If the emission is in or near the direction of the Earth, its apparent speed – even the purely transverse component – is higher than c. Such situations are now regularly observed with telescopes. Ref. 296

Note that to a second observer at the *entrance* of the tunnel, the apparent speed of the car *moving away* is given by

$$v_{leav} = \frac{vc}{c + v} \,, \tag{123}$$

which is *never* higher than $c/2$. In other words, objects are never seen departing with more than half the speed of light.

The story has a final twist. We have just seen that motion faster than light can be observed in several ways. But could an *object* moving faster than light be observed at all? Surprisingly, it could be observed only in rather unusual ways. First of all, since such an imaginary object, usually called a *tachyon*, moves faster than light, we can never see it approaching. If it can be seen at all, a tachyon can only be seen departing. Seeing a tachyon would be similar to hearing a supersonic jet. Only *after* a tachyon has passed nearby, assuming that it is visible in daylight, could we notice it. We would first see a flash of light, corresponding to the bang of a plane passing with supersonic speed. Then we would see *two* images of the tachyon, appearing somewhere in space and departing

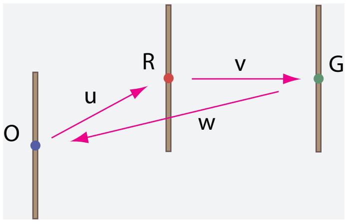

FIGURE 173 If O's stick is parallel to R's and R's is parallel to G's, then O's stick and G's stick are not

in opposite directions, as can be deduced from Figure 172. Even if one of the two images were approaching us, it would be getting fainter and smaller. This is, to say the least, rather unusual behaviour. Moreover, if you wanted to look at a tachyon at night, illuminating it with a torch, you would have to turn your head in the direction opposite to the arm with the torch! This requirement also follows from the space-time diagram: can you see why? Nobody has ever seen such phenomena. Tachyons, if they existed, would be strange objects: they would accelerate when they lose energy, a zero-energy tachyon would be the fastest of all, with infinite speed, and the direction of motion of a tachyon depends on the motion of the observer. No object with these properties has ever been observed. Worse, as we just saw, tachyons would seem to appear from nothing, defying laws of conservation; and note that, just as tachyons cannot be seen in the usual sense, they cannot be touched either, since both processes are due to electromagnetic interactions, as we will see later in our ascent of Motion Mountain. Tachyons therefore cannot be objects in the usual sense. In the second part of our adventure we will show that quantum theory actually *rules out* the existence of (real) tachyons. However, quantum theory also *requires* the existence of 'virtual' tachyons, as we will discover.

Challenge 630 e

Ref. 297

Page 324

Parallel to parallel is not parallel – Thomas rotation

Relativity has strange consequences indeed. Any two observers can keep a stick parallel to the other's, even if they are in motion with respect to each other. But strangely, given a a chain of sticks for which any two adjacent ones are parallel, the first and the last sticks will *not* generally be parallel. In particular, they *never* will be if the motions of the various observers are in different directions, as is the case when the velocity vectors form a loop.

The simplest set-up is shown in Figure 173. In special relativity, a general concatenation of pure boosts does not give a pure boost, but a boost plus a rotation. As a result, the endpoints of chains of parallel sticks are usually not parallel.

Ref. 298

An example of this effect appears in rotating motion. If we walk in a fast circle holding a stick, always keeping the stick parallel to the direction it had just before, at the end of the circle the stick will have an angle with respect to the original direction. Similarly, the axis of a rotating body circling a second body will *not* be pointing in the same direction after one turn. This effect is called *Thomas precession*, after Llewellyn Thomas, who discovered it in 1925, a full 20 years after the birth of special relativity. It had escaped the attention of dozens of other famous physicists. Thomas precession is important in the inner working of atoms; we will return to it in a later section of our adventure. These surprising phenomena are purely relativistic, and are thus measurable *only* in the case of speeds comparable to that of light.

A NEVER-ENDING STORY – TEMPERATURE AND RELATIVITY

The literature on temperature is confusing. Albert Einstein and Wolfgang Pauli agreed on the following result: the temperature T seen by an observer moving with speed v is related to the temperature T_0 measured by the observer at rest with respect to the heat bath via

$$T = T_0 \sqrt{1 - v^2/c^2} \ . \qquad (124)$$

A moving observer thus always measures lower values than a resting one.

In 1908, Max Planck used this expression, together with the corresponding transformation for heat, to deduce that the entropy is invariant under Lorentz transformations. Being the discoverer of the Boltzmann constant k, Planck proved in this way that the constant is a relativistic invariant.

Not all researchers agree on the expression. Others maintain that T and T_0 should be interchanged in the temperature transformation. Also, powers other than the simple Ref. 299 square root have been proposed. The origin of these discrepancies is simple: temperature is only defined for equilibrium situations, i.e. for baths. But a bath for one observer is not a bath for the other. For low speeds, a moving observer sees a situation that is *almost* a heat bath; but at higher speeds the issue becomes tricky. Temperature is deduced from the speed of matter particles, such as atoms or molecules. For moving observers, there is no good way to measure temperature. The naively measured temperature value even depends on the energy range of matter particles that is measured! In short, thermal equilibrium is not an observer-invariant concept. Therefore, *no* temperature transformation formula is correct. (With certain additional assumptions, Planck's expression does seem to hold, however.) In fact, there are not even any experimental observations that would allow such a formula to be checked. Realizing such a measurement is a challenge for future experimenters – but not for relativity itself.

RELATIVISTIC MECHANICS

Because the speed of light is constant and velocities do not add up, we need to rethink the definitions of mass, momentum and energy. We thus need to recreate a theory of mechanics from scratch.

MASS IN RELATIVITY

Page 77 In Galilean physics, the mass ratio between two bodies was defined using collisions; it was given by the negative inverse of the velocity change ratio

$$\frac{m_2}{m_1} = -\frac{\Delta v_1}{\Delta v_2} \ . \qquad (125)$$

However, experiments show that the expression must be different for speeds near that of Challenge 631 ny light. In fact, experiments are not needed: thinking alone can show this. Can you do so?

There is only one solution to this problem. The two Galilean conservation theorems
Ref. 300 $\sum_i m_i \mathbf{v}_i = \text{const}$ for momentum and $\sum_i m_i = \text{const}$ for mass have to be changed into

$$\sum_i \gamma_i m_i \mathbf{v}_i = \text{const} \tag{126}$$

and

$$\sum_i \gamma_i m_i = \text{const} . \tag{127}$$

These expressions, which will remain valid throughout the rest of our ascent of Motion Mountain, imply, among other things, that teleportation is *not* possible in nature. (Can
Challenge 632 n you confirm this?) Obviously, in order to recover Galilean physics, the relativistic correction (factors) γ_i have to be almost equal to 1 for everyday velocities, that is, for velocities nowhere near the speed of light.

Even if we do not know the value of the relativistic correction factor, we can deduce it from the collision shown in Figure 174.

In the first frame of reference (A) we have $\gamma_v m v = \gamma_V M V$ and $\gamma_v m + m = \gamma_V M$. From the observations of the second frame of reference (B) we
Challenge 633 e deduce that V composed with V gives v, in other words, that

$$v = \frac{2V}{1 + V^2/c^2} . \tag{128}$$

When these equations are combined, the relativistic correction γ is found to depend on the magnitude of the velocity v through

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} . \tag{129}$$

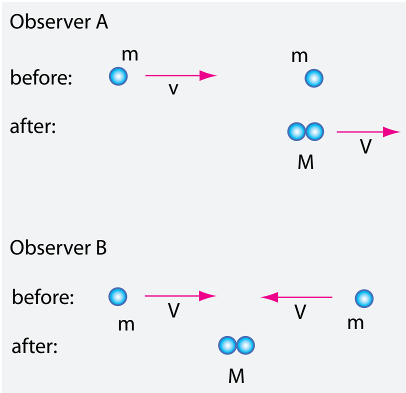

FIGURE 174 An inelastic collision of two identical particles seen from two different inertial frames of reference

With this expression, and a generalization of the situation of Galilean physics, the *mass* ratio between two colliding particles is defined as the ratio

$$\frac{m_1}{m_2} = -\frac{\Delta(\gamma_2 v_2)}{\Delta(\gamma_1 v_1)} . \tag{130}$$

(We do not give here the generalized mass definition, mentioned in the chapter on Ga-
Page 80 lilean mechanics, that is based on acceleration ratios, because it contains some subtleties, which we will discover shortly.) The correction factors γ_i ensure that the mass defined by this equation is the same as the one defined in Galilean mechanics, and that it is the same for all types of collision a body may have.* In this way, mass remains a quantity

Challenge 634 e * The results below also show that $\gamma = 1 + T/mc^2$, where T is the kinetic energy of a particle.

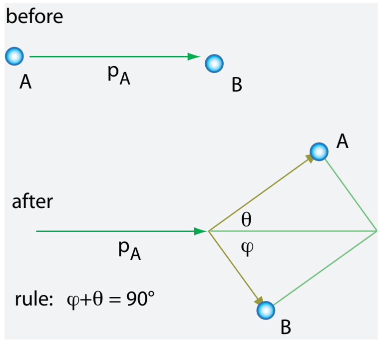

FIGURE 175 A useful rule for playing non-relativistic snooker

characterizing the difficulty of accelerating a body, and it can still be used for *systems* of bodies as well.

Following the example of Galilean physics, we call the quantity

$$\mathbf{p} = \gamma m \mathbf{v} \tag{131}$$

the (*linear*) *relativistic* (*three-*) *momentum* of a particle. Again, the total momentum is a *conserved* quantity for any system not subjected to external influences, and this conservation is a direct consequence of the way mass is defined.

For low speeds, or $\gamma \approx 1$, relativistic momentum is the same as that of Galilean physics, and is proportional to velocity. But for high speeds, momentum increases faster than velocity, tending to infinity when approaching light speed.

Why relativistic snooker is more difficult

There is a well-known property of collisions between a moving sphere or particle and a resting one of the *same mass* that is important when playing snooker, pool or billiards. After such a collision, the two spheres will depart at a *right angle* from each other, as shown in Figure 175.

However, experiments show that the right angle rule does *not* apply to relativistic collisions. Indeed, using the conservation of momentum and a bit of dexterity you can calculate that

Challenge 635 e

$$\tan\theta \tan\varphi = \frac{2}{\gamma+1} , \tag{132}$$

where the angles are defined in Figure 176. It follows that the sum $\varphi + \theta$ is *smaller* than a right angle in the relativistic case. Relativistic speeds thus completely change the game of snooker. Indeed, every accelerator physicist knows this: for electrons or protons, these angles can easily be deduced from photographs taken in cloud chambers, which show the tracks left by particles when they move through them. All such photographs confirm the above expression. In fact, the shapes of detectors are chosen according to expression (132), as sketched in Figure 176. If the formula – and relativity – were wrong, most of these detectors would not work, as they would miss most of the particles after the collision. In

Ref. 267

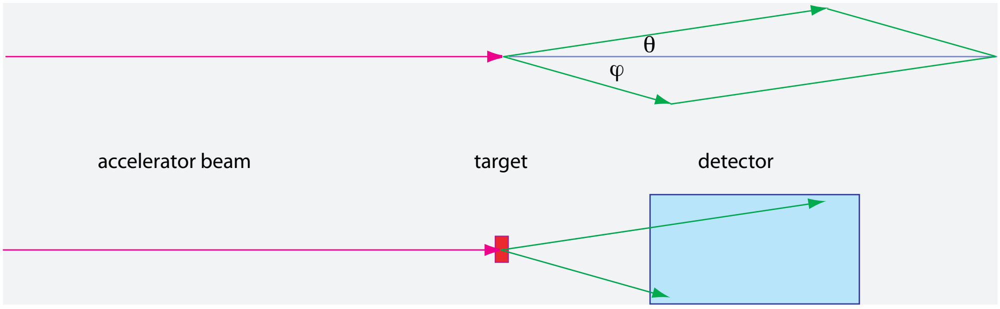

FIGURE 176 The dimensions of detectors in particle accelerators are based on the relativistic snooker angle rule

fact, these experiments also prove the formula for the composition of velocities. Can you show this? Challenge 636 ny

Mass is concentrated energy

Let us go back to the collinear and inelastic collision of Figure 174. What is the mass M of the final system? Calculation shows that Challenge 637 n

$$M/m = \sqrt{2(1+\gamma_v)} > 2 \ . \tag{133}$$

In other words, the mass of the final system is *larger* than the sum of the two original masses m. In contrast to Galilean mechanics, the sum of all masses in a system is *not* a conserved quantity. Only the sum $\sum_i \gamma_i m_i$ of the corrected masses is conserved.

Relativity provides the solution to this puzzle. Everything falls into place if, for the *energy* E of an object of mass m and velocity v, we use the expression

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \ , \tag{134}$$

applying it both to the total system and to each component. The conservation of the corrected mass can then be read as the conservation of energy, simply without the factor c^2. In the example of the two identical masses sticking to each other, the two particles are thus each described by mass and energy, and the resulting system has an energy E given by the sum of the energies of the two particles. In particular, it follows that the energy E_0 of a body *at rest* and its mass m are related by

$$E_0 = mc^2 \ , \tag{135}$$

which is perhaps the most beautiful and famous discovery of modern physics. Since c^2 is so large, we can say that *mass is concentrated energy*. In other words, special relativity says that every mass has energy, and that every form of energy in a system has mass. Increasing the energy of a system increases its mass, and decreasing the energy content decreases the mass. In short, if a bomb explodes inside a closed box, the mass, weight and momentum of the box are the same before and after the explosion, but the combined mass of the debris inside the box will be *smaller* than before. All bombs – not only nuclear ones – thus take their energy from a reduction in mass. In addition, every action of a system – such a caress, a smile or a look – takes its energy from a reduction in mass.

The kinetic energy T is thus given by

$$T = \gamma mc^2 - mc^2 = \frac{1}{2}mv^2 + \frac{1 \cdot 3}{2 \cdot 4} m \frac{v^4}{c^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} m \frac{v^6}{c^4} + \ldots \qquad (136)$$

Challenge 638 e (using the binomial theorem) which reduces to the Galilean value only for low speeds.

The mass–energy equivalence $E = \gamma mc^2$ implies that taking *any* energy from matter results in a mass decrease. When a person plays the piano, thinks or runs, its mass decreases. When a cup of tea cools down or when a star shines, its mass decreases. The mass–energy equivalence pervades all of nature.

By the way, we should be careful to distinguish the transformation of *mass* into energy from the transformation of *matter* into energy. The latter is much more rare. Can you give some examples? Challenge 639 n

The mass–energy relation (134) means the death of many science fiction fantasies. It implies that there are *no* undiscovered sources of energy on or near Earth. If such sources existed, they would be measurable through their mass. Many experiments have looked for, and are still looking for, such effects with a negative result. There is no free energy in nature.*

The mass–energy relation $m = E_0/c^2$ also implies that one needs about 90 thousand million kJ (or 21 thousand million kcal) to increase one's weight by one single gram. Of course, dieticians have slightly different opinions on this matter! In fact, humans do get their everyday energy from the material they eat, drink and breathe by reducing its combined mass before expelling it again. However, this *chemical mass defect* appearing when fuel is burned cannot yet be measured by weighing the materials before and after the reaction: the difference is too small, because of the large conversion factor involved. Indeed, for any chemical reaction, bond energies are about 1 aJ (6 eV) per bond; this gives a weight change of the order of one part in 10^{10}, too small to be measured by weighing people or determining mass differences between food and excrement. Therefore, for everyday chemical processes mass can be taken to be constant, in accordance with Galilean physics.

The mass–energy equivalence has been confirmed by all experiments performed so far. The measurement is simplest for the *nuclear mass defect*. The most precise experiment, from 2005, confirmed the mass–energy relation to more than 6 digits, by comparing the masses difference of nuclei before and after neutron capture one one hand, and emitted gamma ray energy on the other hand. Ref. 301

Modern methods of mass measurement of single molecules have even made it possible to measure the *chemical* mass defect, by comparing the mass of a single molecule with that of its constituent atoms. David Pritchard's group has developed so-called *Penning traps*, which allow masses to be determined from the measurement of frequencies;

* There may be two extremely diluted, yet undiscovered, form of energy, called *dark matter* and (confusingly) *dark energy*, scattered throughout the universe. They are deduced from (quite difficult) mass measurements. The issue has not yet been finally resolved. Page 450

the attainable precision of these cyclotron resonance experiments is sufficient to confirm
Ref. 302 $\Delta E_0 = \Delta mc^2$ for chemical bonds. In the future, increased precision will even allow bond energies to be determined in this way with precision. Since binding energy is often radiated as light, we can say that these modern techniques make it possible to *weigh* light.

Thinking about light and its mass was the basis for Einstein's first derivation of the mass–energy relation. When an object emits two equal light beams in opposite directions, its energy decreases by the emitted amount. Since the two light beams are equal in energy and momentum, the body does not move. If we describe the same situation from the
Challenge 640 e viewpoint of a moving observer, we see again that the *rest energy* of the object is

$$E_0 = mc^2 . \tag{137}$$

In summary, all physical processes, including collisions, need relativistic treatment whenever the energy involved is a sizeable fraction of the rest energy.

Every energy increase produces a mass increase. Therefore also heating a body makes it heavier. However, this effect is so small that nobody has measured it up to this day. It is a challenge for experiments of the future to do this one day.

How are energy and momentum related? The definitions of momentum (131) and en-
Challenge 641 e ergy (134) lead to two basic relations. First of all, their magnitudes are related by

$$m^2c^4 = E^2 - p^2c^2 \tag{138}$$

for all relativistic systems, be they objects or, as we will see below, radiation. For the momentum *vector* we get the other important relation

$$\mathbf{p} = \frac{E}{c^2}\mathbf{v} , \tag{139}$$

Challenge 642 e which is equally valid for *any* type of moving energy, be it an object or a beam or a pulse of radiation.* We will use both relations often in the rest of our ascent of Motion Mountain, including the following discussion.

Collisions, virtual objects and tachyons

We have just seen that in relativistic collisions the conservation of total energy and momentum are intrinsic consequences of the definition of mass. Let us now have a look at collisions in more detail, using these new concepts. A *collision* is a process, i.e. a series of events, for which

— the total momentum before the interaction and after the interaction is the same;
— the momentum is exchanged in a small region of space-time;
— for small velocities, the Galilean description is valid.

In everyday life an *impact*, i.e. a short-distance interaction, is the event at which both objects change momentum. But the two colliding objects are located at *different* points
Ref. 303 when this happens. A collision is therefore described by a space-time diagram such as the

* In 4-vector notation, we can write $v/c = \mathbf{P}/P_0$, where $P_0 = E/c$.

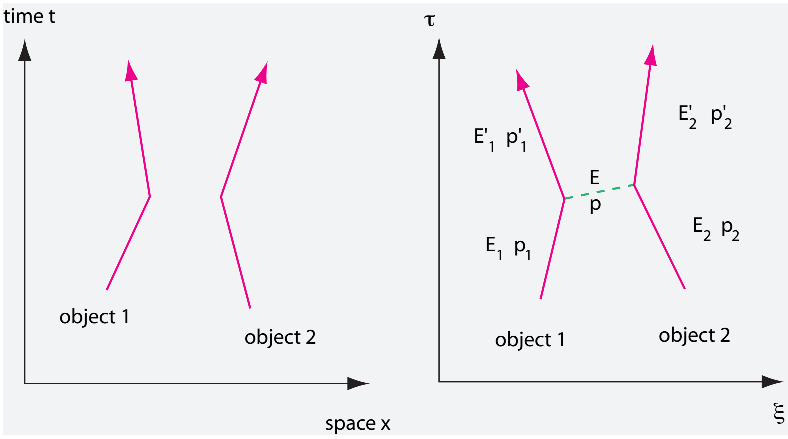

FIGURE 177 Space-time diagram of a collision for two observers

left-hand one in Figure 177, reminiscent of the Orion constellation. It is easy to check that the process described by such a diagram is a collision according to the above definition. Challenge 643 e

The right-hand side of Figure 177 shows the same process seen from another, Greek, frame of reference. The Greek observer says that the first object has changed its momentum *before* the second one. That would mean that there is a short interval when momentum and energy are *not* conserved!

The only way to make sense of the situation is to assume that there is an exchange of a third object, drawn with a dotted line. Let us find out what the properties of this object are. If we give numerical subscripts to the masses, energies and momenta of the two bodies, and give them a prime after the collision, the unknown mass m obeys Challenge 644 e

$$m^2c^4 = (E_1 - E_1')^2 - (p_1 - p_1')^2c^2 = 2m_1^2c^4 - 2E_1E_1'\left(\frac{1 - v_1v_1'}{c^2}\right) < 0 \,. \qquad (140)$$

This is a strange result, because it means that the unknown mass is an *imaginary* number!!* On top of that, we also see directly from the second graph that the exchanged object moves faster than light. It is a *tachyon*, from the Greek ταχύς 'rapid'. In other words, collisions involve motion that is faster than light! We will see later that collisions are indeed the *only* processes where tachyons play a role in nature. Since the exchanged objects appear only during collisions, never on their own, they are called *virtual* objects, to distinguish them from the usual, *real* objects, which can move freely without restriction.** We will study their properties later on, when we come to discuss quantum theory.

* It is usual to change the mass–energy and mass–momentum relation of tachyons to $E = \pm mc^2/\sqrt{v^2/c^2 - 1}$ and $p = \pm mv/\sqrt{v^2/c^2 - 1}$; this amounts to a redefinition of m. After the redefinition, tachyons have *real* mass. The energy and momentum relations show that tachyons lose energy and momentum when they get faster. (Provocatively, a single tachyon in a box could provide us with all the energy we need.) Both signs for the energy and momentum relations must be retained, because otherwise the equivalence of all inertial observers would not be generated. Tachyons thus do not have a minimum energy or a minimum momentum.
** More precisely, a virtual particle does not obey the relation $m^2c^4 = E^2 - p^2c^2$, valid for real particles.

In nature, a tachyon is always a virtual object. Real objects are always *bradyons* – from the Greek βραδύς 'slow' – or objects moving slower than light. Note that tachyons, despite their high velocity, do not allow the transport of energy faster than light; and that they do not violate causality if and only if they are emitted or absorbed with equal probability. Challenge 645 ny Can you confirm all this?

When we study quantum theory, we will also discover that a general contact interaction between objects is described not by the exchange of a *single* virtual object, but by a continuous *stream* of virtual particles. For standard collisions of everyday objects, the interaction turns out to be electromagnetic. In this case, the exchanged particles are virtual photons. In other words, when one hand touches another, when it pushes a stone, or when a mountain supports the trees on it, streams of virtual photons are continuously Page 750 exchanged.

There is an additional secret hidden in collisions. In the right-hand side of Figure 177, the tachyon is emitted by the first object and absorbed by the second one. However, it Challenge 646 n is easy to imagine an observer for which the opposite happens. In short, the direction of travel of a tachyon depends on the observer! In fact, this is a hint about *antimatter*. In space-time diagrams, matter and antimatter travel in opposite directions. Also the connection between relativity and antimatter will become more apparent in quantum Page 778 theory.

Systems of particles – no centre of mass

Relativity also forces us to eliminate the cherished concept of *centre of mass*. We can see this already in the simplest example possible: that of two equal objects colliding.

Figure 178 shows that from the viewpoint in which one of two colliding particles is at rest, there are at least three different ways to define the centre of mass. In other words, the Ref. 304 centre of mass is not an observer-invariant concept. We can deduce from the figure that the concept only makes sense for systems whose components move with *small* velocities relative to each other. For more general systems, centre of mass is not uniquely definable. Will this hinder us in our ascent? No. We are more interested in the motion of single particles than that of composite objects or systems.

Why is most motion so slow?

For most everyday systems, the time intervals measured by two different observers are practically equal; only at large relative speeds, typically at more than a few per cent of the speed of light, is there a noticeable difference. Most such situations are microscopic. We have already mentioned the electrons inside a television tube or inside a particle accelerator. The particles making up cosmic radiation are another example: their high energy has produced many of the mutations that are the basis of evolution of animals and plants on this planet. Later we will discover that the particles involved in radioactivity are also relativistic.

But why don't we observe any rapid *macroscopic* bodies? Moving bodies, including observers, with relativistic velocities have a property not found in everyday life: when they are involved in a collision, part of their energy is converted into new matter via $E = \gamma mc^2$. In the history of the universe this has happened so many times that practically all the bodies still in relativistic motion are microscopic particles.

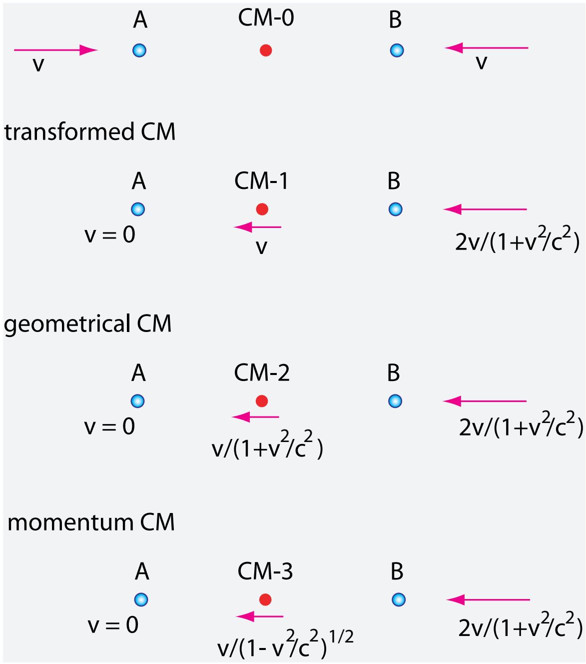

FIGURE 178 There is no way to define a relativistic centre of mass

A second reason for the disappearance of rapid relative motion is radiation damping. Can you imagine what happens to charges during collisions, or in a bath of light? Challenge 647 n

In short, almost all matter in the universe moves with small velocity relative to other matter. The few known counter-examples are either very old, such as the quasar jets mentioned above, or stop after a short time. The huge energies necessary for macroscopic relativistic motion are still found in supernova explosions, but they cease to exist after only a few weeks. In summary, the universe is mainly filled with slow motion because it is *old*. We will determine its age shortly. Page 457

The history of the mass–energy equivalence formula of De Pretto and Einstein

Albert Einstein took several months after his first paper on special relativity to deduce the expression

$$E = \gamma m c^2 \tag{141}$$

which is often called the most famous formula of physics. He published it in a second, separate paper towards the end of 1905. Arguably, the formula could have been discovered thirty years earlier, from the theory of electromagnetism. Ref. 256

In fact, at least one person did deduce the result before Einstein. In 1903 and 1904, *before* Einstein's first relativity paper, a little-known Italian engineer, Olinto De Pretto, was the first to calculate, discuss and publish the formula $E = mc^2$.* It might well be that

* Umberto Bartocci, mathematics professor of the University of Perugia in Italy, published the details of

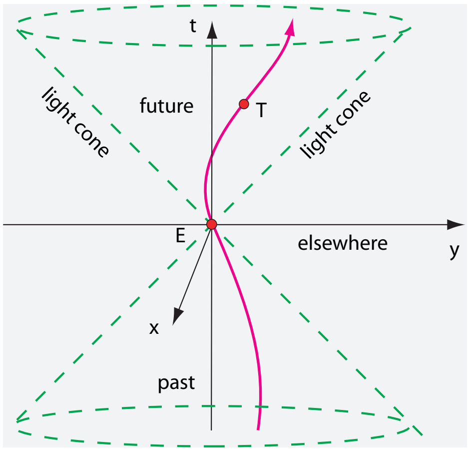

FIGURE 179 The space-time diagram of a moving object T

Einstein got the idea for the formula from De Pretto, possibly through Einstein's friend Michele Besso or other Italian-speaking friends he met when he visited his parents, who were living in Italy at the time. Of course, the value of Einstein's efforts is not diminished by this.

Ref. 305 In fact, a similar formula had also been deduced in 1904 by Friedrich Hasenöhrl and published again in Annalen der Physik in 1905, before Einstein, though with an incorrect numerical prefactor, due to a calculation mistake. The formula $E = mc^2$ is also part of several expressions in two publications in 1900 by Henri Poincaré. The real hero in the story might well be Tolver Preston, who discussed the equivalence of mass and energy already in 1875, in his book *Physics of the Ether*. The mass–energy equivalence was thus indeed floating in the air, only waiting to be discovered.

In the 1970s, there was a similar story: a simple relation between the gravitational acceleration and the temperature of the vacuum was discovered. The result had been waiting to be discovered for over 50 years. Indeed, a number of similar, anterior results were found in the libraries. Could other simple relations be hidden in modern physics waiting to be found? Challenge 648 n

4-vectors

To describe motion consistently for *all* observers, we have to introduce some new quantities. First of all, motion of particles is seen as a sequence of events. To describe events with precision, we use event coordinates, also called *4-coordinates*. These are written as

$$\mathbf{X} = (ct, \mathbf{x}) = (ct, x, y, z) = X^i \ . \tag{142}$$

In this way, an event is a point in four-dimensional space-time, and is described by four coordinates. The coordinates are called the zeroth, namely time $X^0 = ct$, the first, usually

this surprising story in several papers. The full account is found in his book Umberto Bartocci, *Albert Einstein e Olinto De Pretto: la vera storia della formula più famosa del mondo*, Ultreja, 1998.

called $X^1 = x$, the second, $X^2 = y$, and the third, $X^3 = z$. One can then define a *distance d* between events as the length of the difference vector. In fact, one usually uses the square of the length, to avoid those unwieldy square roots. In special relativity, the magnitude ('squared length') of a vector is always defined through

$$\mathbf{XX} = {X_0}^2 - {X_1}^2 - {X_2}^2 - {X_3}^2 = ct^2 - x^2 - y^2 - z^2 = X_a X^a = \eta_{ab} X^a X^b = \eta^{ab} X_a X_b \ . \tag{143}$$

In this equation we have introduced for the first time two notations that are useful in relativity. First of all, we automatically sum over repeated indices. Thus, $X_a X^a$ means the sum of all products $X_a X^a$ as a ranges over all indices. Secondly, for every 4-vector $\mathbf{X}$ we distinguish two ways to write the coordinates, namely coordinates with superscripts and coordinates with subscripts. (In three dimensions, we only use subscripts.) They are related by the following general relation

$$X_a = \eta_{ab} X^b = (ct, -x, -y, -z) \ , \tag{144}$$

where we have introduced the so-called *metric* η^{ab}, an abbreviation of the matrix*

$$\eta^{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{145}$$

Don't panic: this is all, and it won't get more difficult! We now go back to physics.

The magnitude of a position or distance vector, also called the space-time *interval*, is essentially the proper time times c. The *proper time* is the time shown by a clock moving in a straight line and with constant velocity from the starting point to the end point in space-time. The difference from the usual 3-vectors is that the magnitude of the interval can be positive, negative or even zero. For example, if the start and end points in space-time require motion with the speed of light, the proper time is zero (this is required for
Page 304
null vectors). If the motion is slower than the speed of light, the squared proper time is positive and the distance is timelike. For negative intervals and thus imaginary proper times, the distance is spacelike.** A simplified overview is given by Figure 179.

Now we are ready to calculate and measure motion in four dimensions. The measurements are based on one central idea. We cannot define the velocity of a particle as the derivative of its coordinates with respect to time, since time and temporal sequences depend on the observer. The solution is to define all observables with respect to the just-mentioned *proper time* τ, which is defined as the time shown by a clock attached to the object. In relativity, motion and change are always measured with respect to clocks attached to the moving system. In particular, the *relativistic velocity* or *4-velocity* $\mathbf{U}$ of a body is

* Note that 30 % of all physics textbooks use the negative of η as the metric, the so-called *spacelike convention*, and thus have opposite signs in this definition. In this text, as in 70 % of all physics texts, we use the *timelike convention*.

** In the latter case, the negative of the magnitude, which is a positive number, is called the squared *proper distance*. The proper distance is the length measured by an odometer as the object moves along.

thus defined as the rate of change of the event coordinates or *4-coordinates* $\mathbf{X} = (ct, \mathbf{x})$ with respect to proper time, i.e. as

$$\mathbf{U} = \mathrm{d}\mathbf{X}/\mathrm{d}\tau \; . \tag{146}$$

The coordinates **X** are measured in the coordinate system defined by the inertial observer chosen. The value of the velocity **U** depends on the observer or coordinate system used; so the velocity depends on the observer, as it does in everyday life. Using $\mathrm{d}t = \gamma\, \mathrm{d}\tau$ and thus

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{, where as usual} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \; , \tag{147}$$

we get the relation with the 3-velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$:

$$u^0 = \gamma c \; , \; u^i = \gamma v_i \quad \text{or} \quad \mathbf{U} = (\gamma c, \gamma \mathbf{v}) \; . \tag{148}$$

For small velocities we have $\gamma \approx 1$, and then the last three components of the 4-velocity are those of the usual, Galilean 3-velocity. For the magnitude of the 4-velocity **U** we find $\mathbf{UU} = U_a U^a = \eta_{ab} U^a U^b = c^2$, which is therefore independent of the magnitude of the 3-velocity **v** and makes it a timelike vector, i.e. a vector *inside* the light cone.*

Note that the magnitude of a 4-vector can be zero even though all its components are different from zero. Such a vector is called *null*. Which motions have a null velocity vector? Challenge 650 n

Similarly, the *relativistic acceleration* or *4-acceleration* **B** of a body is defined as

$$\mathbf{B} = \mathrm{d}\mathbf{U}/\mathrm{d}\tau = \mathrm{d}^2\mathbf{X}/\mathrm{d}\tau^2 \; . \tag{150}$$

Using $\mathrm{d}\gamma/\mathrm{d}\tau = \gamma \mathrm{d}\gamma/\mathrm{d}t = \gamma^4 \mathbf{va}/c^2$, we get the following relations between the four components of **B** and the 3-acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$: Ref. 306

$$B^0 = \gamma^4 \frac{\mathbf{va}}{c} \quad , \quad B^i = \gamma^2 a_i + \gamma^4 \frac{(\mathbf{va}) v_i}{c^2} \; . \tag{151}$$

The magnitude b of the 4-acceleration is easily found via $\mathbf{BB} = \eta_{cd} B^c B^d = -\gamma^4 (a^2 + \gamma^2 (\mathbf{va})^2/c^2) = -\gamma^6 (a^2 - (\mathbf{v} \times \mathbf{a})^2/c^2)$. Note that it does depend on the value of the

* In general, a *4-vector* is defined as a quantity (h_0, h_1, h_2, h_3), which transforms as

$$\begin{aligned} h_0' &= \gamma_V (h_0 - h_1 V/c) \\ h_1' &= \gamma_V (h_1 - h_0 V/c) \\ h_2' &= h_2 \\ h_3' &= h_3 \end{aligned} \tag{149}$$

when changing from one inertial observer to another moving with a relative velocity V in the x direction; the corresponding generalizations for the other coordinates are understood. This relation allows one to deduce the transformation laws for any 3-vector. Can you deduce the velocity composition formula (109) from this definition, applying it to 4-velocity? Challenge 649 n

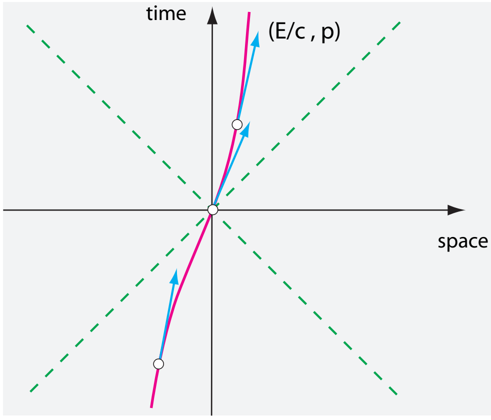

FIGURE 180 Energy–momentum is tangent to the world line

3-acceleration **a**. The magnitude of the 4-acceleration is also called the *proper* acceleration because $\mathbf{B}^2 = -a^2$ if $v = 0$. (What is the connection between 4-acceleration and 3-acceleration for an observer moving with the same speed as the object?) We note that 4-acceleration lies *outside* the light cone, i.e. that it is a spacelike vector, and that $\mathbf{BU} = \eta_{cd} B^c U^d = 0$, which means that the 4-acceleration is always perpendicular to the 4-velocity.* We also note that accelerations, in contrast to velocities, cannot be called relativistic: the difference between b_i and a_i, or between their two magnitudes, does not depend on the value of a_i, but only on the value of the speed v. In other words, accelerations require relativistic treatment only when the involved velocities are relativistic. If the velocities involved are low, even the highest accelerations can be treated with Galilean methods.

Challenge 651 n

When the acceleration **a** is parallel to the velocity **v**, we get $B = \gamma^3 a$; when **a** is perpendicular to **v**, as in circular motion, we get $B = \gamma^2 a$. We will use this result below.

Page 338

4-MOMENTUM

To describe motion, we also need the concept of momentum. The *4-momentum* is defined as

$$\mathbf{P} = m\mathbf{U} \qquad (154)$$

* Similarly, the *relativistic jerk* or *4-jerk* **J** of a body is defined as

$$\mathbf{J} = \mathrm{d}\mathbf{B}/\mathrm{d}\tau = \mathrm{d}^2\mathbf{U}/\mathrm{d}\tau^2 \; . \qquad (152)$$

Challenge 652 e

For the relation with the 3-jerk $\mathbf{j} = \mathrm{d}\mathbf{a}/\mathrm{d}t$ we then get

$$\mathbf{J} = (J^0, J^i) = \left(\frac{\gamma^5}{c}(\mathbf{jv} + a^2 + 4\gamma^2 \frac{(\mathbf{va})^2}{c^2}) \, , \, \gamma^3 j_i + \frac{\gamma^5}{c^2}((\mathbf{jv})v_i + a^2 v_i + 4\gamma^2 \frac{(\mathbf{va})^2 v_i}{c^2} + 3(\mathbf{va})a_i) \right) \qquad (153)$$

Challenge 653 ny

which we will use later on. Surprisingly, **J** does not vanish when **j** vanishes. Why not?

and is therefore related to the 3-momentum $\mathbf{p}$ by

$$\mathbf{P} = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p}) . \tag{155}$$

For this reason 4-momentum is also called the *energy–momentum 4-vector*. In short, *the 4-momentum of a body is given by mass times 4-displacement per proper time*. This is the simplest possible definition of momentum and energy. The concept was introduced by Max Planck in 1906. The energy–momentum 4-vector, also called *momenergy*, like the 4-velocity, is *tangent* to the world line of a particle. This connection, shown in Figure 180, follows directly from the definition, since

$$(E/c, \mathbf{p}) = (\gamma mc, \gamma m\mathbf{v}) = m(\gamma c, \gamma\mathbf{v}) = m(\mathrm{d}t/\mathrm{d}\tau, \mathrm{d}\mathbf{x}/\mathrm{d}\tau) . \tag{156}$$

The (square of the) length of momenergy, namely $\mathbf{PP} = \eta_{ab}P^aP^b$, is by definition the same for all inertial observers; it is found to be

$$E^2/c^2 - p^2 = m^2c^2 , \tag{157}$$

thus confirming a result given above. We have already mentioned that energies or situations are called *relativistic* if the kinetic energy $T = E - E_0$ is not negligible when compared to the rest energy $E_0 = mc^2$. A particle whose kinetic energy is much higher than its rest mass is called *ultrarelativistic*. Particles in accelerators or in cosmic rays fall into this category. (What is their energy–momentum relation?) Challenge 654 n

In contrast to Galilean mechanics, relativity implies an absolute zero for the energy. One cannot extract more energy than mc^2 from a system of mass m. In particular, a zero value for potential energy is fixed in this way. In short, relativity shows that energy is bounded from below.

Note that by the term 'mass' m we always mean what is sometimes called the *rest mass*. This name derives from the bad habit of many science fiction and secondary-school books of calling the product γm the *relativistic mass*. Workers in the field usually (but not unanimously) reject this concept, as did Einstein himself, and they also reject the often-heard expression that '(relativistic) mass increases with velocity'. Relativistic mass and energy would then be two words for the same concept: this way to talk is at the level of the tabloid press. Ref. 307

Not all Galilean energy contributes to mass. Potential energy in an outside field does not. Relativity forces us into precise energy bookkeeping. 'Potential energy' in relativity is an abbreviation for 'energy reduction of the outside field'.

Can you show that for two particles with momenta P_1 and P_2, one has $P_1P_2 = m_1E_2 = M_2E_1 = c^2\gamma v_{12}m_1m_2$, where v_{12} is their relative velocity? Challenge 655 n

4-force

The 4-force $\mathbf{K}$ is defined as

$$\mathbf{K} = \mathrm{d}\mathbf{P}/\mathrm{d}\tau = m\mathbf{B} . \tag{158}$$

Ref. 306, Ref. 308

Therefore force remains equal to mass times acceleration in relativity. From the definition of **K** we deduce the relation with 3-force $\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}t = m\mathrm{d}(\gamma\mathbf{v})/\mathrm{d}t$, namely*

$$\mathbf{K} = (K^0, K^i) = (\gamma^4 m\mathbf{va}/c, \gamma^2 m a_i + \gamma^4 v_i \frac{m\mathbf{va}}{c^2}) = (\frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \gamma\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}) = (\gamma\frac{\mathbf{fv}}{c}, \gamma\mathbf{f}) \ . \qquad (159)$$

Challenge 657 e

The 4-force, like the 4-acceleration, is orthogonal to the 4-velocity. The meaning of the zeroth component of the 4-force can easily be discerned: it is the *power* required to accelerate the object. One has $\mathbf{KU} = c^2 \mathrm{d}m/\mathrm{d}\tau = \gamma^2(\mathrm{d}E/\mathrm{d}t - \mathbf{fv})$: this is the proper rate at which the internal energy of a system increases. The product **KU** vanishes only for rest-mass-conserving forces. Particle collisions that lead to reactions do not belong to this class. In everyday life, the rest mass is preserved, and then one gets the Galilean expression $\mathbf{fv} = \mathrm{d}E/\mathrm{d}t$.

Rotation in relativity

If at night we turn around our own axis while looking at the sky, the stars move with a velocity much higher than that of light. Most stars are masses, not images. Their speed should be limited by that of light. How does this fit with special relativity?

This example helps to clarify in another way what the limit velocity actually is. Physically speaking, a rotating sky does *not* allow superluminal energy transport, and thus does not contradict the concept of a limit speed. Mathematically speaking, the speed of light limits relative velocities *only* between objects that come *near* to each other, as shown on the left of Figure 181. To compare velocities of distant objects is only possible if all velocities involved are constant in time; this is not the case in the present example. The differential version of the Lorentz transformations make this point particularly clear. In many general cases, relative velocities of *distant* objects can be higher than the speed of light. We encountered one example earlier, when discussing the car in the tunnel, and we will encounter a few more examples shortly.

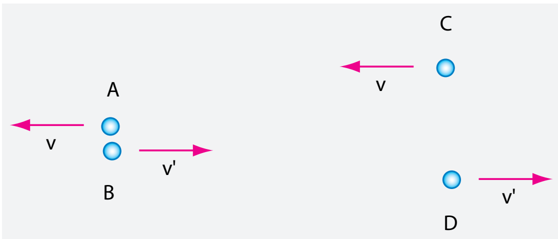

FIGURE 181 On the definition of relative velocity

Page 315
Page 345

With this clarification, we can now briefly consider *rotation* in relativity. The first question is how lengths and times change in a rotating frame of reference. You may want to check that an observer in a rotating frame agrees with a non-rotating colleague on the radius of a rotating body; however, both find that the rotating body, even if it is rigid, has a circumference *different* from the one it had before it started rotating. Sloppily speaking, the value of π *changes* for rotating observers. The ratio between the circumference c and the radius r turns out to be $c/r = 2\pi\gamma$: it increases with rotation speed. This

Challenge 658 e

Challenge 659 e

* Some authors define 3-force as $\mathrm{d}\mathbf{p}/\mathrm{d}\tau$; then **K** looks slightly different. In any case, it is important to note that in relativity, 3-force $\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}t$ is indeed proportional to 3-acceleration **a**; however, force and acceleration are not parallel to each other. In fact, for rest-mass-preserving forces one finds $\mathbf{f} = \gamma m\mathbf{a} + (\mathbf{fv})\mathbf{v}/c^2$. In contrast, in relativity 3-momentum is *not* proportional to 3-velocity, although it is parallel to it.

Challenge 656 n

Ref. 309 counter-intuitive result is often called *Ehrenfest's paradox*. Among other things, it shows that space-time for an observer on a rotating disc is *not* the Minkowski space-time of special relativity.

Rotating bodies behave strangely in many ways. For example, one gets into trouble when one tries to synchronize clocks mounted on a rotating circle,as shown in Figure 182 If one starts synchronizing the clock at O_2 with that at O_1, and so on, continuing up to clock O_n, one finds that the last clock is *not* synchronized with the first. This result reflects the change in circumference just mentioned. In fact, a careful study shows that the measurements of length and time intervals lead all observers O_k to conclude that they live in a rotating space-time. Rotating discs can thus be used as an introduction to general relativity, where this curvature and its effects form the central topic. More about this in the next chapter.

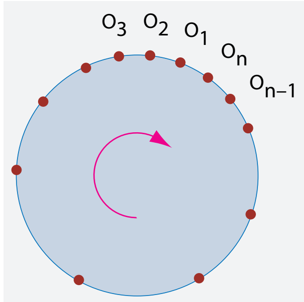

FIGURE 182 Observers on a rotating object

Is angular velocity limited? Yes: the tangential speed in an inertial frame of reference cannot exceed that of light. The limit thus depends on the *size* of the body in question.
Challenge 660 ny That leads to a neat puzzle: can one *see* objects rotating very rapidly?

We mention that 4-angular momentum is defined naturally as

$$l^{ab} = x^a p^b - x^b p^a \; . \qquad (160)$$

In other words, 4-angular momentum is a *tensor*, not a vector, as shown by its two indices.
Challenge 661 ny Angular momentum is conserved in special relativity. The moment of inertia is naturally defined as the proportionality factor between angular velocity and angular momentum.

Obviously, for a rotating particle, the rotational energy is part of the rest mass. You may
Challenge 662 ny want to calculate the fraction for the Earth and the Sun. It is not large. By the way, how
Challenge 663 ny would you determine whether a microscopic particle, too small to be seen, is rotating?

In relativity, rotation and translation combine in strange ways. Imagine a cylinder in uniform rotation along its axis, as seen by an observer at rest. As Max von Laue has discussed, the cylinder will appear *twisted* to an observer moving along the rotation axis.
Challenge 664 e Can you confirm this?

Here is a last puzzle about rotation. Velocity is relative; this means that the measured
Challenge 665 ny value depends on the observer. Is this the case also for angular velocity?

Wave motion

In Galilean physics, a wave is described by a wave vector and a frequency. In special relativity, the two are combined in the wave 4-vector, given by

$$\mathbf{L} = \frac{1}{\lambda}\left(\frac{\omega}{c}, \mathbf{n}\right) \; , \qquad (161)$$

where λ is the wavelength, ω the wave velocity, and **n** the normed direction vector. Suppose an observer with 4-velocity **U** finds that a wave **L** has frequency ν. Show that

$$\nu = \mathbf{LU} \tag{162}$$

Challenge 666 ny must be obeyed. Interestingly, the wave velocity ω transforms in a different way than Ref. 261 particle velocity except in the case $\omega = c$. Also the aberration formula for wave motion Challenge 667 ny differs from that for particles, except in the case $\omega = c$.

The action of a free particle – how do things move?

If we want to describe relativistic motion of a free particle in terms of an extremal principle, we need a definition of the action. We already know that physical action is a measure of the change occurring in a system. Page 178 For an inertially moving or free particle, the only change is the ticking of its proper clock. As a result, the action of a free particle will be proportional to the elapsed proper time. In order to get the standard unit of energy times time, or Js, for the action, the first guess for the action of a free particle is

$$S = -mc^2 \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \,, \tag{163}$$

where τ is the proper time along its path. This is indeed the correct expression. It implies conservation of (relativistic) energy and momentum, as the change in proper time is maximal for straight-line motion with constant velocity. Can you confirm this? Challenge 668 ny Indeed, in nature, all particles move in such a way that their proper time is maximal. In other words, we again find that in nature things change as little as possible. Nature is like a wise old man: its motions are as slow as possible. If you prefer, every change is maximally effective. As we mentioned before, Bertrand Russell called this the *law of cosmic laziness*.

The expression (163) for the action is due to Max Planck. In 1906, by exploring it in detail, he found that the quantum of action $\hbar$, which he had discovered together with the Boltzmann constant, is a relativistic invariant (like the Boltzmann constant k). Can you imagine how he did this? Challenge 669 ny

The action can also be written in more complex, seemingly more frightening ways. These equivalent ways to write it are particularly appropriate to prepare for general relativity:

$$S = \int L \,\mathrm{d}t = -mc^2 \int_{t_1}^{t_2} \frac{1}{\gamma} \,\mathrm{d}t = -mc \int_{\tau_1}^{\tau_2} \sqrt{u_a u^a} \,\mathrm{d}\tau = -mc \int_{s_1}^{s_2} \sqrt{\eta^{ab} \frac{\mathrm{d}x_a}{\mathrm{d}s} \frac{\mathrm{d}x_b}{\mathrm{d}s}} \,\mathrm{d}s \,, \tag{164}$$

where s is some arbitrary, but monotonically increasing, function of τ, such as τ itself. As usual, the *metric* $\eta^{\alpha\beta}$ of special relativity is

$$\eta^{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{165}$$

You can easily confirm the form of the action (164) by deducing the equation of motion in the usual way.
Challenge 670 ny

In short, nature is not in a hurry: every object moves in a such way that its own clock shows the *longest* delay possible, compared with any alternative motion nearby.* This general principle is also valid for particles under the influence of gravity, as we will see in the section on general relativity, and for particles under the influence of electric or magnetic interactions. In fact, it is valid in all cases of (macroscopic) motion found in nature. For the moment, we just note that the longest proper time is realized when the difference between kinetic and potential energy is minimal. (Can you confirm this?) For the Galilean case, the longest proper time thus implies the smallest average difference between the two energy types. We thus recover the principle of least action in its Galilean formulation.
Challenge 672 ny

Page 178

Earlier on, we saw that the action measures the change going on in a system. Special relativity shows that nature minimizes change by maximizing proper time. In nature, *proper time is always maximal*. In other words, things move along paths of *maximal ageing*. Can you explain why 'maximal ageing' and 'cosmic laziness' are equivalent?
Challenge 673 ny

We thus again find that nature is the opposite of a Hollywood film: nature changes in the most economical way possible. The deeper meaning of this result is left to your personal thinking: enjoy it!

Conformal transformations – why is the speed of light constant?

The distinction between space and time in special relativity depends on the observer. On the other hand, all inertial observers agree on the position, shape and orientation of the light cone at a point. Thus, in the theory of relativity, the light cones are the basic physical 'objects'. Given the importance of light cones, we might ask if inertial observers are the only ones that observe the same light cones. Interestingly, it turns out that *other* observers do as well.

The first such category of observers are those using units of measurement in which all time and length intervals are multiplied by a *scale factor* λ. The transformations among these points of view are given by

$$x_a \mapsto \lambda x_a \tag{166}$$

and are called *dilations*.

A second category of additional observers are found by applying the so-called *special conformal transformations*. These are compositions of an *inversion*

$$x_a \mapsto \frac{x_a}{x^2} \tag{167}$$

with a *translation* by a vector b_a, namely

$$x_a \mapsto x_a + b_a \ , \tag{168}$$

* If neutrinos were massless, the action (164) would not be applicable for them. Why? Can you find an alternative for this (admittedly academic) case?
Challenge 671 ny

and a second inversion. Thus the special conformal transformations are

$$x_a \mapsto \frac{x_a + b_a x^2}{1 + 2b_a x^a + b^2 x^2} \quad \text{or} \quad \frac{x_a}{x^2} \mapsto \frac{x_a}{x^2} + b_a \ . \tag{169}$$

These transformations are called *conformal* because they do not change angles of (infinitesimally) small shapes, as you may want to check. They therefore leave the *form* (of infinitesimally small objects) unchanged. For example, they transform infinitesimal circles into infinitesimal circles. They are called *special* because the *full* conformal group includes the dilations and the inhomogeneous Lorentz transformations as well.*

Challenge 674 ny

Note that the way in which special conformal transformations leave light cones invariant is rather subtle.

Challenge 676 ny

Since dilations do not commute with time translations, there is no conserved quantity associated with this symmetry. (The same is true of Lorentz boosts.) In contrast, rotations and spatial translations do commute with time translations and thus do lead to conserved quantities.

In summary, vacuum is conformally invariant – in the special sense just mentioned – and thus also dilation invariant. This is another way to say that vacuum alone is not sufficient to define lengths, as it does not fix a scale factor. As we would expect, matter is necessary to do so. Indeed, (special) conformal transformations are not symmetries of situations containing matter. Only vacuum is conformally invariant; nature as a whole is not.

However, conformal invariance, or the invariance of light cones, is sufficient to allow velocity measurements. Conformal invariance is also *necessary* for velocity measurements, as you might want to check.

Challenge 677 ny

We have seen that conformal invariance implies inversion symmetry: that is, that the large and small scales of a vacuum are related. This suggest that the constancy of the speed of light is related to the existence of inversion symmetry. This mysterious connection gives us a glimpse of the adventures we will encounter in the third part of our ascent of Motion Mountain. Conformal invariance turns out to be an important property that will lead to some incredible insights.**

Challenge 675 ny

* The set of all *special* conformal transformations forms a group with four parameters; adding dilations and the inhomogeneous Lorentz transformations one gets fifteen parameters for the *full* conformal group. The conformal group is locally isomorphic to SU(2,2) and to the simple group SO(4,2): these concepts are explained in Appendix D. Note that all this is true only for *four* space-time dimensions; in *two* dimensions – the other important case, especially in string theory – the conformal group is isomorphic to the group of arbitrary analytic coordinate transformations, and is thus infinite-dimensional.

Page 1210

** The conformal group does not appear only in the kinematics of special relativity: it is the symmetry group of all physical interactions, such as electromagnetism, provided that all the particles involved have zero mass, as is the case for the photon. A field that has mass cannot be conformally invariant; therefore conformal invariance is not an exact symmetry of all of nature. Can you confirm that a mass term $m\varphi^2$ in a Lagrangian is not conformally invariant?

Challenge 678 ny

However, since all particles observed up to now have masses that are many orders of magnitude smaller than the Planck mass, it can be said that they have almost vanishing mass; conformal symmetry can then be seen as an *approximate* symmetry of nature. In this view, all massive particles should be seen as small corrections, or perturbations, of massless, i.e. conformally invariant, fields. Therefore, for the construction of a fundamental theory, conformally invariant Lagrangians are often assumed to provide a good starting

Accelerating observers

So far, we have only studied what inertial, or free-flying, observers say to each other when they talk about the same observation. For example, we saw that moving clocks always run slow. The story gets even more interesting when one or both of the observers are accelerating.

One sometimes hears that special relativity cannot be used to describe accelerating observers. That is wrong, just as it is wrong to say that Galilean physics cannot be used for accelerating observers. Special relativity's only limitation is that it cannot be used in non-flat, i.e. curved, space-time. Accelerating bodies do exist in flat space-times, and therefore they can be discussed in special relativity.

Ref. 310 As an appetizer, let us see what an accelerating, Greek, observer says about the clock of an inertial, Roman, one, and vice versa. Assume that the Greek observer, shown in Figure 183, moves along the path $\mathbf{x}(t)$, as observed by the inertial Roman one. In general, the Roman/Greek clock rate ratio is given by $\Delta\tau/\Delta t = (\tau_2 - \tau_1)/(t_2 - t_1)$. Here the Greek coordinates are constructed with a simple procedure: take the two sets of events defined by $t = t_1$ and $t = t_2$, and let τ_1 and τ_2 be the points where these sets intersect the time axis of the Greek observer.* We assume that the Greek observer is inertial and moving with velocity v as observed by the Roman one. The clock ratio of a Greek observer is then given by

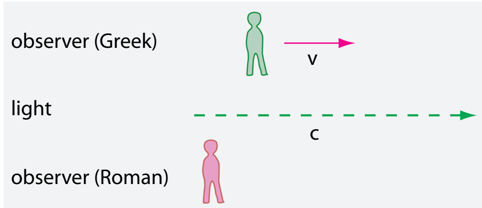

FIGURE 183 The simplest situation for an inertial and an accelerated observer

$$\frac{\Delta\tau}{\Delta t} = \frac{d\tau}{dt} = \sqrt{1 - v^2/c^2} = \frac{1}{\gamma_v} \; , \qquad (170)$$

Challenge 679 ny a formula we are now used to. We find again that moving clocks run slow.

Ref. 310 For accelerated motions, the differential version of the above reasoning is necessary. The Roman/Greek clock rate ratio is again $d\tau/dt$, and τ and $\tau + d\tau$ are calculated in the same way from the times t and $t + dt$. Assume again that the Greek observer moves along the path $\mathbf{x}(t)$, as measured by the Roman one. We find directly that

$$\tau = t - \mathbf{x}(t)\mathbf{v}(t)/c^2 \qquad (171)$$

and thus

$$\tau + d\tau = (t + dt) - [\mathbf{x}(t) - dt\mathbf{v}(t)][\mathbf{v}(t) + dt\mathbf{a}(t)]/c^2 \; . \qquad (172)$$

Together, these equations yield

$$\text{‘}d\tau/dt\text{’} = \gamma_v(1 - \mathbf{v}\mathbf{v}/c^2 - \mathbf{x}\mathbf{a}/c^2) \; . \qquad (173)$$

approximation.

* These sets form what mathematicians call *hypersurfaces*.

This shows that accelerated clocks can run *fast* or slow, depending on their position **x** and the sign of their acceleration **a**. There are quotes in the above equation because we can see directly that the Greek observer notes

$$\text{`}\mathrm{d}t/\mathrm{d}\tau\text{'} = \gamma_v \ , \tag{174}$$

which is *not* the inverse of equation (173). This difference becomes most apparent in the simple case of two clocks with the same velocity, one of which has a constant acceleration g towards the origin, whereas the other moves inertially. We then have

$$\text{`}\mathrm{d}\tau/\mathrm{d}t\text{'} = 1 + gx/c^2 \tag{175}$$

and

$$\text{`}\mathrm{d}t/\mathrm{d}\tau\text{'} = 1 \ . \tag{176}$$

We will discuss this situation shortly. But first we must clarify the concept of acceleration.

Acceleration for inertial observers

Accelerations behave differently from velocities under change of viewpoint. Let us first take the simple case in which the object and two inertial observers all move along the x-axis. If the Roman inertial observer measures an acceleration $a = \mathrm{d}v/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2$, and the Greek observer, also inertial, measures an acceleration $\alpha = \mathrm{d}\omega/\mathrm{d}\tau = \mathrm{d}^2\xi/\mathrm{d}\tau^2$, we
Ref. 262 get

$$\gamma_v^3 a = \gamma_\omega^3 \alpha \ . \tag{177}$$

This relation shows that accelerations are *not* Lorentz invariant, unless the velocities are small compared to the speed of light. This is in contrast to our everyday experience, where accelerations are independent of the speed of the observer.

Expression (177) simplifies if the accelerations are measured at a time t at which ω vanishes – i.e. if they are measured by the so-called *comoving* inertial observer. In that case the acceleration relation is given by

$$a_\mathrm{c} = a\gamma_v^3 \tag{178}$$

and the acceleration $a_\mathrm{c} = \alpha$ is also called proper acceleration, as its value describes what the Greek, comoving observer *feels*: proper acceleration describes the experience of being pushed into the back of the accelerating seat.

In general, the observer's speed and the acceleration are not parallel. We can calculate
Ref. 311 how the value of 3-acceleration **a** measured by a general inertial observer is related to the value $\mathbf{a}_\mathrm{c}$ measured by the comoving observer using expressions (151) and (149). We get the generalization of (178):

$$\mathbf{v}\mathbf{a}_\mathrm{c} = \mathbf{v}\mathbf{a}\gamma_v^3 \tag{179}$$

and

$$\mathbf{a} = \frac{1}{\gamma_v^2}\left(\mathbf{a}_c - \frac{(1-\gamma_v)(\mathbf{v}\mathbf{a}_c)\mathbf{v}}{v^2} - \frac{\gamma_v(\mathbf{v}\mathbf{a}_c)\mathbf{v}}{c^2}\right) . \tag{180}$$

Squaring yields the relation

$$a^2 = \frac{1}{\gamma_v^4}\left(a_c^2 - \frac{(\mathbf{a}_c\mathbf{v})^2}{c^2}\right) \tag{181}$$

Page 329
which we know already in a slightly different form. It shows (again) that the comoving or proper 3-acceleration is always larger than the 3-acceleration measured by an outside inertial observer. The faster the outside inertial observer is moving, the smaller the acceleration he observes. Acceleration is not a relativistic invariant. The expression also shows that whenever the speed is perpendicular to the acceleration, a boost yields a factor γ_v^2, whereas a speed parallel to the acceleration gives the already mentioned γ_v^3 dependence.

Challenge 680 e

We see that acceleration complicates many issues, and it requires a deeper investigation. To keep matters simple, from now on we only study *constant* accelerations. Interestingly, this situation serves also as a good introduction to black holes and, as we will see shortly, to the universe as a whole.

Page 484

Accelerating frames of reference

How do we check whether we live in an inertial frame of reference? Let us first define the term. An *inertial frame* (*of reference*) has two defining properties. First, lengths and distances measured with a ruler are described by Euclidean geometry. In other words, rulers behave as they do in daily life. In particular, distances found by counting how many rulers (rods) have to be laid down end to end to reach from one point to another – the so-called *rod distances* – behave as in everyday life. For example, they obey Pythagoras' theorem in the case of right-angled triangles. Secondly, the speed of light is constant. In other words, any two observers in that frame, independent of their time and of the position, make the following observation: the ratio c between twice the rod distance between two points and the time taken by light to travel from one point to the other and back is always the same.

Equivalently, an inertial frame is one for which all clocks always remain synchronized and whose geometry is Euclidean. In particular, in an inertial frame all observers at fixed coordinates always remain at rest with respect to each other. This last condition is, however, a more general one. There are other, non-inertial, situations where this is still the case.

Non-inertial frames, or *accelerating frames*, are a useful concept in special relativity. In fact, we all live in such a frame. We can use special relativity to describe it in the same way that we used Galilean physics to describe it at the beginning of our journey.

A general *frame of reference* is a continuous set of observers remaining at rest with respect to each other. Here, 'at rest with respect to each other' means that the time for a light signal to go from one observer to another and back again is constant over time, or equivalently, that the rod distance between the two observers is constant. Any frame of reference can therefore also be called a *rigid* collection of observers. We therefore note that a general frame of reference is *not* the same as a set of coordinates; the latter is usu-

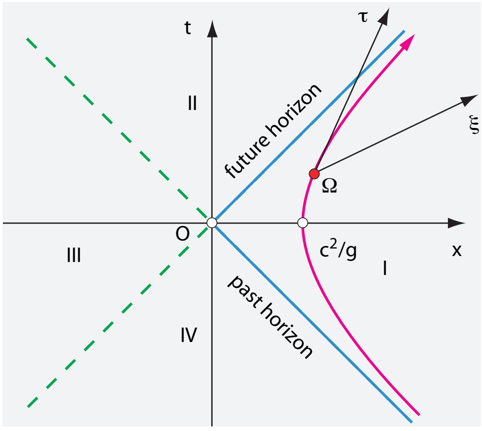

FIGURE 184 The hyperbolic motion of an rectilinearly, uniformly accelerating observer Ω

ally *not* rigid. If all the rigidly connected observers have constant coordinate values, we speak of a *rigid coordinate system*. Obviously, these are the most useful when it comes to describing accelerating frames of reference.*

Ref. 313 Note that if two observers both move with a velocity **v**, as measured in some *inertial* frame, they observe that they are at rest with respect to each other *only* if this velocity is *constant*. Again we find, as above, that two people tied to each other by a rope, and at a distance such that the rope is under tension, will see the rope break (or hang loose) if they accelerate together to (or decelerate from) relativistic speeds in precisely the same way. Relativistic acceleration requires careful thinking. Challenge 681 ny

An observer who always *feels* the *same* force on his body is called *uniformly* accelerating. More precisely, a uniformly accelerating observer is an observer whose acceleration at every moment, measured by the inertial frame with respect to which the observer is at rest *at that moment*, always has the same value **B**. It is important to note that uniform acceleration is *not* uniformly accelerating when always observed from the *same* inertial frame. This is an important difference from the Galilean case.

For uniformly accelerated motion in the sense just defined, we need

$$\mathbf{B} \cdot \mathbf{B} = -g^2 \qquad (182)$$

Ref. 314 where g is a constant independent of t. The simplest case is uniformly accelerating motion that is also *rectilinear*, i.e. for which the acceleration **a** is parallel to **v** at one instant of time and (therefore) for all other times as well. In this case we can write, using 3-vectors, Challenge 682 ny

Ref. 312 * There are essentially only two other types of rigid coordinate frames, apart from the inertial frames:

— The frame $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 (1 + g_k x_k / c^2)^2$ with arbitrary, but constant, acceleration of the origin. The acceleration is $\mathbf{a} = -\mathbf{g}(1 + \mathbf{g}x/c^2)$.
— The uniformly rotating frame $ds^2 = dx^2 + dy^2 + dz^2 + 2\omega(-y\,dx + x\,dy)dt - (1 - r^2\omega^2/c^2)dt$. Here the z-axis is the rotation axis, and $r^2 = x^2 + y^2$.

$$\gamma^3 \mathbf{a} = \mathbf{g} \quad \text{or} \quad \frac{\mathrm{d}\gamma\mathbf{v}}{\mathrm{d}t} = \mathbf{g} \,. \tag{183}$$

Taking the direction we are talking about to be the x-axis, and solving for $v(t)$, we get

$$v = \frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}} \,, \tag{184}$$

where it was assumed that $v(0) = 0$. We note that for small times we get $v = gt$ and for large times $v = c$, both as expected. The momentum of the accelerated observer increases linearly with time, again as expected. Integrating, we find that the accelerated observer moves along the path

Challenge 683 ny

$$x(t) = \frac{c^2}{g}\sqrt{1 + \frac{g^2t^2}{c^2}} \,, \tag{185}$$

where it is assumed that $x(0) = c^2/g$, in order to keep the expression simple. Because of this result, visualized in Figure 184, a rectilinearly and uniformly accelerating observer is said to undergo *hyperbolic* motion. For small times, the world-line reduces to the usual $x = gt^2/2 + x_0$, whereas for large times it is $x = ct$, as expected. The motion is thus uniformly accelerated only for the moving body itself, *not* for an outside observer.

The proper time τ of the accelerated observer is related to the time t of the inertial frame in the usual way by $\mathrm{d}t = \gamma\mathrm{d}\tau$. Using the expression for the velocity $v(t)$ of equation (184) we get*

Ref. 314, Ref. 315

$$t = \frac{c}{g}\sinh\frac{g\tau}{c} \quad \text{and} \quad x = \frac{c^2}{g}\cosh\frac{g\tau}{c} \tag{186}$$

for the relationship between proper time τ and the time t and position x measured by the external, inertial Roman observer. We will encounter this relation again during our study of black holes.

Does all this sound boring? Just imagine accelerating on a motorbike at $g = 10\,\mathrm{m/s^2}$ for the proper time τ of 25 years. That would bring you beyond the end of the known universe! Isn't that worth a try? Unfortunately, neither motorbikes nor missiles that accelerate like this exist, as their fuel tanks would have to be enormous. Can you confirm this?

Challenge 684 n

Ref. 316

* Use your favourite mathematical formula collection – every student should have one – to deduce this. The *hyperbolic sine* and the *hyperbolic cosine* are defined by $\sinh\, y = (\mathrm{e}^y - \mathrm{e}^{-y})/2$ and $\cosh\, y = (\mathrm{e}^y + \mathrm{e}^{-y})/2$. They imply that $\int \mathrm{d}y/\sqrt{y^2+a^2} = \operatorname{arsinh} y/a = \operatorname{Arsh} y/a = \ln(y + \sqrt{y^2+a^2})$.

For uniform acceleration, the coordinates transform as

$$t = \left(\frac{c}{g} + \frac{\xi}{c}\right) \sinh \frac{g\tau}{c}$$
$$x = \left(\frac{c^2}{g} + \xi\right) \cosh \frac{g\tau}{c}$$
$$y = \upsilon$$
$$z = \zeta \, , \qquad (187)$$

where τ now is the time coordinate in the Greek frame. We note also that the space-time interval $\mathrm{d}\sigma$ satisfies

$$\mathrm{d}\sigma^2 = (1 + g\xi/c^2)^2 c^2 \mathrm{d}\tau^2 - \mathrm{d}\xi^2 - \mathrm{d}\upsilon^2 - \mathrm{d}\zeta^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \, , \qquad (188)$$

and since for $\mathrm{d}\tau = 0$ distances are given by Pythagoras' theorem, the Greek reference
Ref. 317 frame is indeed rigid.

After this forest of formulae, let's tackle a simple question, shown in Figure 184. The inertial, Roman observer O sees the Greek observer Ω departing with acceleration g, moving further and further away, following equation (185). What does the Greek observer say about his Roman colleague? With all the knowledge we have now, that is easy to answer. At each point of his trajectory Ω sees that O has the coordinate $\tau = 0$ (can you
Challenge 685 e confirm this?), which means that the distance to the Roman observer, as seen by the Greek
Ref. 318 one, is the same as the space-time interval OΩ. Using expression (185), we see that this is

$$d_{\mathrm{O}\Omega} = \sqrt{\xi^2} = \sqrt{x^2 - c^2 t^2} = c^2/g \, , \qquad (189)$$

which, surprisingly enough, is constant in time! In other words, the Greek observer will observe that he stays at a constant distance from the Roman one, in complete contrast to what the Roman observer says. Take your time to check this strange result in some other way. We will need it again later on, to explain why the Earth does not explode. (Can you
Challenge 686 n guess how that is related to this result?)

The composition theorem for accelerations is more complex than for velocities. The
Ref. 319 best explanation of this was published by Mishra. If we call a_{nm} the acceleration of system n by observer m, we are seeking to express the object acceleration a_{01} as function of the value a_{02} measured by the other observer, the relative acceleration a_{12}, and the proper acceleration a_{22} of the other observer: see Figure 185. Here we will only study one-dimensional situations, where all observers and all objects move along one axis. (For clarity, we also write $v_{11} = v$ and $v_{02} = u$.) In Galilean physics we have the general connec-
Challenge 687 e tion

$$a_{01} = a_{02} - a_{12} + a_{22} \qquad (190)$$

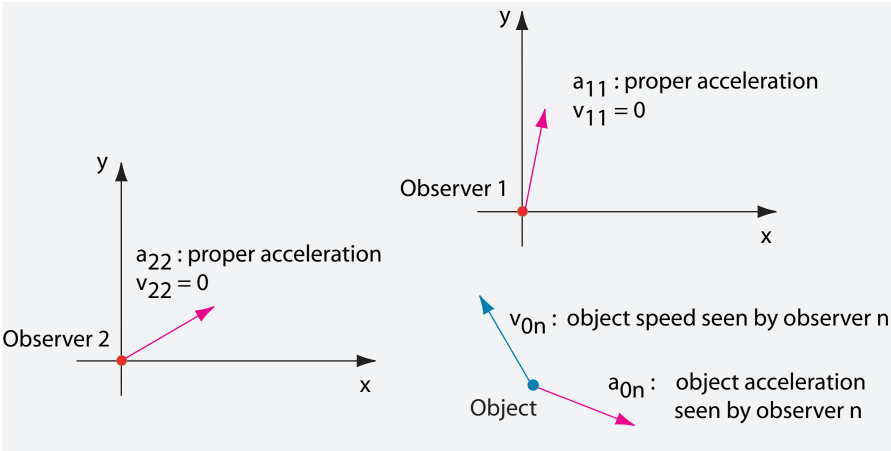

FIGURE 185 The definitions necessary to deduce the composition behaviour of accelerations

because accelerations behave simply. In special relativity, one gets

$$a_{01} = a_{02}\frac{(1-v^2/c^2)^{3/2}}{(1-uv/c^2)^3} - a_{12}\frac{(1-u^2/c^2)(1-v^2/c^2)^{-1/2}}{(1-uv/c^2)^2} + a_{22}\frac{(1-u^2/c^2)(1-v^2/c^2)^{3/2}}{(1-uv/c^2)^3} \qquad (191)$$

Challenge 688 ny

and you might enjoy checking the expression.

Page 319

Can you state how the acceleration ratio enters into the definition of mass in special relativity?

Challenge 689 ny

Event horizons

There are many surprising properties of accelerated motion. Of special interest is the trajectory, in the coordinates ξ and τ of the rigidly accelerated frame, of an object located at the departure point $x = x_0 = c^2/g$ at all times t. One gets the two relations*

Challenge 690 ny

$$\xi = -\frac{c^2}{g}(1 - \operatorname{sech}\frac{g\tau}{c})$$
$$d\xi/d\tau = -c \operatorname{sech}\frac{g\tau}{c}\tanh\frac{g\tau}{c} . \qquad (193)$$

These equations are strange. For large times τ the coordinate ξ approaches the limit value $-c^2/g$ and $d\xi/d\tau$ approaches zero. The situation is similar to that of a car accelerating away from a woman standing on a long road. Seen from the car, the woman moves away; however, after a while, the only thing one notices is that she is slowly approaching the

* The functions appearing above, the *hyperbolic secant* and the *hyperbolic tangent*, are defined using the expressions from the footnote on page 341:

$$\operatorname{sech} y = \frac{1}{\cosh y} \quad \text{and} \quad \tanh y = \frac{\sinh y}{\cosh y} . \qquad (192)$$

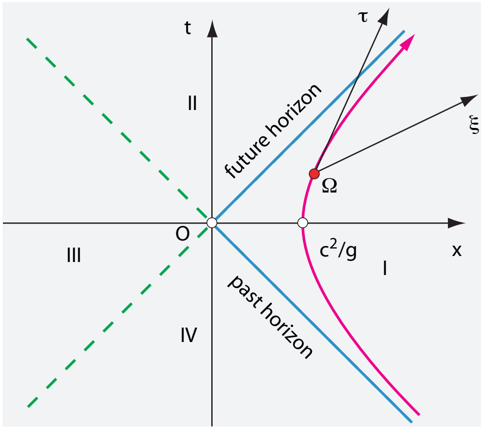

FIGURE 186 Hyperbolic motion and event horizons

horizon. In Galilean physics, both the car driver and the woman on the road see the other person approaching their horizon; in special relativity, only the accelerated observer makes this observation.

A graph of the situation helps to clarify the result. In Figure 186 we can see that light emitted from any event in regions II and III cannot reach the Greek observer. Those events are hidden from him and cannot be observed. Strangely enough, however, light from the Greek observer *can* reach region II. The boundary between the part of space-time that can be observed and the part that cannot is called the *event horizon*. In relativity, event horizons act like one-way gates for light and other signals. For completeness, the graph also shows the past event horizon. Can you confirm that event horizons are *black*? Challenge 691 ny

So, not all events observed in an inertial frame of reference can be observed in a uniformly accelerating frame of reference. Uniformly accelerating frames of reference produce event horizons at a distance $-c^2/g$. For example, a person who is standing can never see further than this distance below his feet.

By the way, is it true that a light beam cannot catch up with an observer in hyperbolic motion, if the observer has a sufficient headstart? Challenge 692 n

Here is a more advanced challenge, which prepares us for general relativity. What is the *shape* of the horizon seen by a uniformly accelerated observer? Challenge 693 n

Acceleration changes colours

We saw earlier that a moving receiver sees different colours from the sender. So far, we discussed this colour shift, or Doppler effect, for inertial motion only. For accelerating frames the situation is even stranger: sender and receiver do not agree on colours even if they are at *rest* with respect to each other. Indeed, if light is emitted in the direction of the acceleration, the formula for the space-time interval gives Ref. 314, Ref. 320

$$d\sigma^2 = \left(1 + \frac{g_0 x}{c^2}\right)^2 c^2 dt^2 \qquad (194)$$

in which g_0 is the proper acceleration of an observer located at $x = 0$. We can deduce in
Challenge 694 ny
a straightforward way that

$$\frac{f_r}{f_s} = 1 - \frac{g_r h}{c^2} = \frac{1}{\left(1 + \frac{g_s h}{c^2}\right)} \qquad (195)$$

where h is the rod distance between the source and the receiver, and where $g_s = g_0/(1 + g_0 x_s/c^2)$ and $g_r = g_0/(1 + g_o x_r/c^2)$ are the proper accelerations measured at the source and at the detector. In short, the frequency of light decreases when light moves in the direction of acceleration. By the way, does this have an effect on the colour of trees along
Challenge 695 n
their vertical extension?

The formula usually given, namely

$$\frac{f_r}{f_s} = 1 - \frac{gh}{c^2} , \qquad (196)$$

is only correct to a first approximation. In accelerated frames of reference, we have to be careful about the meaning of every quantity. For everyday accelerations, however, the
Challenge 696 ny
differences between the two formulae are negligible. Can you confirm this?

Can light move faster than c?

What speed of light does an accelerating observer measure? Using expression (196) above, an accelerated observer deduces that

$$v_{\text{light}} = c\,(1 + \frac{gh}{c^2}) \qquad (197)$$

which is higher than c for light moving in front of or 'above' him, and lower than c for light moving behind or 'below' him. This strange result follows from a basic property of any accelerating frame of reference. In such a frame, even though all observers are at rest with respect to each other, clocks do *not* remain synchronized. This change of the speed of light has also been confirmed by experiment.* Thus, the speed of light is only constant when it is defined as $c = \mathrm{d}x/\mathrm{d}t$, *and* if $\mathrm{d}x$ and $\mathrm{d}t$ are measured with a ruler located at a point *inside* the interval $\mathrm{d}x$ and a clock read off *during* the interval $\mathrm{d}t$. If the speed of light is defined as $\Delta x/\Delta t$, or if the ruler defining distances or the clock measuring times is located away from the propagating light, the speed of light is different from c for accelerating observers! This is the same effect you can experience when you turn around your vertcial axis at night: the star velocities you observe are much higher than the speed of light.

Note that this result does not imply that signals or energy can be moved faster than c.
Challenge 697 n
You may want to check this for yourself.

In fact, all these effects are negligible for distances l that are much less than c^2/a. For an acceleration of $9.5\,\mathrm{m/s^2}$ (about that of free fall), distances would have to be of the order

Page 419
* The propagation delays to be discussed in the chapter on general relativity can be seen as confirmations of this effect.

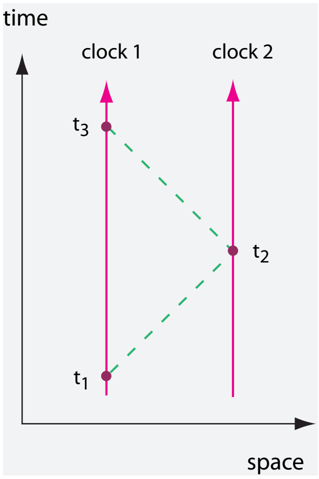

FIGURE 187 Clocks and the measurement of the speed of light as two-way velocity

of one light year, or $9.5 \cdot 10^{12}$ km, in order for any sizable effects to be observed. In short, *c is the speed of light relative to* nearby *matter only.*

By the way, everyday gravity is equivalent to a constant acceleration. So, why then do distant objects, such as stars, not move faster than light, following expression (197)? Challenge 698 n

What is the speed of light?

We have seen that the speed of light, as usually defined, is given by c only if either the observer is inertial or the observer measures the speed of light passing nearby (rather than light passing at a distance). In short, the speed of light has to be measured locally. But this condition does not eliminate all subtleties.

An additional point is often forgotten. Usually, length is measured by the time it takes light to travel. In such a case the speed of light will obviously be constant. But how does one check the constancy? One needs to eliminate length measurements. The simplest way to do this is to reflect light from a mirror, as shown in Figure 187. The constancy of the speed of light implies that if light goes up and down a short straight line, then the clocks at the two ends measure times given by

$$t_3 - t_1 = 2\,(t_2 - t_1)\;. \qquad (198)$$

Here it is assumed that the clocks have been synchronised according to the prescription on page 307. If the factor were not exactly two, the speed of light would not be constant. In fact, all experiments so far have yielded a factor of two, within measurement errors.*

* The subtleties of the one-way and two-way speed of light will remain a point of discussion for a long time. Many experiments are explained and discussed in Ref. 267. Zhang says in his summary on page 171, that the one-way velocity of light is indeed independent of the light source; however, no experiment really shows that it is equal to the two-way velocity. Moreover, most so called 'one-way' experiments are in fact still 'two-way' experiments (see his page 150). Ref. 321

This result is sometimes expressed by saying that it is impossible to measure the *one-way velocity of light*; only the *two-way* velocity of light is measurable. Do you agree? Challenge 699 n

Limits on the length of solid bodies

An everyday solid object breaks when some part of it moves with respect to some other part with more than the speed of sound c of the material.* For example, when an object hits the floor and its front end is stopped within a distance d, the object breaks at the latest when

$$\frac{v^2}{c^2} \geqslant \frac{2d}{l} . \tag{199}$$

In this way, we see that we can avoid the breaking of fragile objects by packing them into foam rubber – which increases the stopping distance – of roughly the same thickness as the object's size. This may explain why boxes containing presents are usually so much larger than their contents!

The fracture limit can also be written in a different way. To avoid breaking, the acceleration a of a solid body with length l must obey

$$l a < c^2 , \tag{200}$$

where c is the speed of sound, which is the speed limit for the material parts of solids. Let us now repeat the argument in relativity, using the speed of light instead of that of sound. Ref. 322 Imagine accelerating the front of a *solid* body with some *proper* acceleration a. The back end cannot move with an acceleration α equal or larger than infinity, or if one prefers, it cannot move with more than the speed of light. A quick check shows that therefore the length l of a solid body must obey Challenge 700 n

$$l\alpha < c^2/2 , \tag{201}$$

where c is now the speed of light. The speed of light thus limits the size of solid bodies. For example, for $9.8\,\mathrm{m/s^2}$, the acceleration of good motorbike, this expression gives a length limit of 9.2 Pm, about a light year. Not a big restriction: most motorbikes are shorter.

However, there are other, more interesting situations. The highest accelerations achievable today are produced in particle accelerators. Atomic nuclei have a size of a few femtometres. Can you deduce at which energies they break when smashed together in an accelerator? Challenge 701 ny In fact, inside a nucleus, the nucleons move with accelerations of the order of $v^2/r \approx \hbar^2/m^2r^3 \approx 10^{31}\,\mathrm{m/s^2}$; this is one of the highest values found in nature.

Note that Galilean physics and relativity produce a similar conclusion: a limiting speed, be it that of sound or that of light, makes it impossible for solid bodies to be *rigid*. When we push one end of a body, the other end always moves a little bit later.

A puzzle: does the speed limit imply a relativistic 'indeterminacy relation'

$$\Delta l \; \Delta a \leqslant c^2 \tag{202}$$

* The (longitudinal) speed of sound is about 5.9 km/s for glass, iron or steel; about 4.5 km/s for gold; and about 2 km/s for lead. Other sound speeds are given on page 208.

Page 1089 for the length and acceleration indeterminacies?

What does all this mean for the size of elementary particles? Take two electrons a distance d apart, and call their size l. The acceleration due to electrostatic repulsion then leads to an upper limit for their size given by Challenge 702 ny

$$l < \frac{4\pi\varepsilon_0 c^2 d^2 m}{e^2} . \qquad (203)$$

The nearer electrons can get, the smaller they must be. The present experimental limit gives a size smaller than 10^{-19} m. Can electrons be exactly point-like? We will come back to this question during our study of general relativity and quantum theory.

Special relativity in four sentences

This section of our ascent of Motion Mountain can be quickly summarized.

— All (free floating) observers find that there is a unique, perfect velocity in nature, namely a common maximum energy velocity, which is realized by massless radiation such as light or radio signals, but cannot be achieved by material systems.
— Therefore, even though space-time is the same for every observer, times and lengths vary from one observer to another, as described by the Lorentz transformations (113) and (114), and as confirmed by experiment.
— Collisions show that a maximum speed implies that mass is concentrated energy, and that the total energy of a body is given by $E = \gamma mc^2$, as again confirmed by experiment.
— Applied to accelerated objects, these results lead to numerous counter-intuitive consequences, such as the twin paradox, the appearance of event horizons and the appearance of short-lived tachyons in collisions.

Special relativity shows that motion, though limited in speed, is relative, defined using the propagation of light, conserved, reversible and deterministic.

Could the speed of light vary?

The speed of massless light is the limit speed. Assuming that all light is indeed massless, could the speed of light still change from place to place, or as time goes by? This tricky question still makes a fool out of many physicists. The first answer is usually a loud: 'Yes, of course! Just look at what happens when the value of c is changed in formulae.' (In fact, there have even been attempts to build 'variable speed of light theories'.) However, this often-heard statement is wrong.

Since the speed of light enters into our definition of time and space, it thus enters, even if we do not notice it, into the construction of all rulers, all measurement standards and all measuring instruments. Therefore there is no way to detect whether the value actually varies. No imaginable experiment could detect a variation of the limit speed, as the limit speed is the basis for all measurements. 'That is intellectual cruelty!', you might say. 'All experiments show that the speed of light is invariant; we had to swallow one counter-intuitive result after another to accept the constancy of the speed of light, and now we are supposed to admit that there is no other choice?' Yes, we are. That is the irony of progress in physics. The observer-invariance of the speed of light is counter-intuitive Challenge 703 n

and astonishing when compared to the lack of observer-invariance at everyday, Galilean speeds. But had we taken into account that every speed measurement is – whether we like it or not – a comparison with the speed of light, we would not have been astonished by the constancy of the speed of light; rather, we would have been astonished by the strange properties of *small* speeds.

In short, there is in principle no way to check the invariance of a standard. To put it another way, the truly surprising aspect of relativity is not the invariance of c; it is the disappearance of c from the formulae of everyday motion.

What happens near the speed of light?

As one approaches the speed of light, the quantities in the Lorentz transformation diverge. A division by zero is impossible: indeed, neither masses nor observers can move at the speed of light. However, this is only half the story.

No observable actually diverges in nature. Approaching the speed of light as nearly as possible, even special relativity breaks down. At extremely large Lorentz contractions, there is no way to ignore the curvature of space-time; indeed, gravitation has to be taken into account in those cases. Near horizons, there is no way to ignore the fluctuations of speed and position; quantum theory has to be taken into account there. The exploration of these two limitations define the next two stages of our ascent of Motion Mountain.

At the start of our adventure, during our exploration of Galilean physics, once we had defined the basic concepts of velocity, space and time, we turned our attention to gravitation. The invariance of the speed of light has forced us to change these basic concepts. We now return to the study of gravitation in the light of this invariance.

Bibliography

246 Aristotle, On sense and the sensible, section 1, part 1, 350 bce. Cited in Jean-Paul Dumont, *Les écoles présocratiques*, Folio Essais, Gallimard, p. 157, 1991. Cited on page 284.

247 The history of the measurement of the speed of light can be found in chapter 19 of the text by Francis A. Jenkins & Harvey E. White, *Fundamentals of Optics*, McGraw-Hill, New York, 1957. Cited on page 284.

248 On the way to perform such measurements, see Sydney G. Brewer, *Do-it-yourself Astronomy*, Edinburgh University Press, 1988. Kepler himself never measured the distances of planets to the Sun, but only *ratios* of planetary distances. The parallax of the Sun from two points of the Earth is at most 8.79″; it was first measured in the eighteenth century. Cited on page 286.

249 Aristarchos, On the sizes and the distances of the Sun and the Moon, *c.* 280 bce, in Michael J. Crowe, *Theories of the World From Antiquity to the Copernican Revolution*, Dover, 1990. Cited on page 286.

250 J. Frercks, Creativity and technology in experimentation: Fizeau's terrestrial determination of the speed of light, *Centaurus* **42**, pp. 249–287, 2000. See also the beautiful website on reconstrutions of historical science experiments at http://www.uni-oldenburg.de/histodid/forschung/nachbauten. Cited on page 286.

251 The way to make pictures of light pulses with an ordinary photographic camera, without any electronics, is described by M.A. Duguay & A.T. Mattick, Ultrahigh speed photography of picosecond light pulses and echoes, *Applied Optics* **10**, pp. 2162–2170, 1971. The picture on page 287 is taken from it. Cited on page 287.

252 You can learn the basics of special relativity with the help of the web, without referring to any book, using the http://physics.syr.edu/research/relativity/RELATIVITY.html web page as a starting point. This page mentions most of the English-language relativity resources available on the web. Links in other languages can be found with search engines. Cited on page 288.

253 Observations of gamma ray bursts show that the speed of light does not depend on the lamp speed to within one part in 10^{20}, as shown by K. Brecher, *Bulletin of the American Physical Society* **45**, 2000. He assumed that both sides of the burster emit light. The large speed difference and the pulse sharpness then yield this result. See also his older paper K. Brecher, Is the speed of light independent of the source?, *Physics Letters* **39**, pp. 1051–1054, Errata 1236, 1977. Measuring the light speed from rapidly moving stars is another way. Some of these experiments are not completely watertight, however. There is a competing theory of electrodynamics, due to Ritz, which maintains that the speed of light is c only when measured with respect to the source; The light from stars, however, passes through the atmosphere, and its speed might thus be reduced to c.

The famous experiment with light emitted from rapid pions at CERN is not subject to this criticism. It is described in T. Alväger, J.M. Bailey, F.J.M. Farley, J. Kjellman & I. Wallin, Test of the second postulate of relativity in the GeV region, *Physics Letters* **12**, pp. 260–262, 1964. See also T. Alväger & al., Velocity of high-energy gamma rays, *Arkiv för Fysik* **31**, pp. 145–157, 1965.

Another precise experiment at extreme speeds is described by G.R. Kalbfleisch, N. Baggett, E.C. Fowler & J. Alspector, Experimental comparison of neutrino, antineutrino, and muon velocities, *Physical Review Letters* **43**, pp. 1361–1364, 1979. Cited on page 288.

254 See e.g. C. Will, *Theory and Experiment in Gravitational Physics*, Revised edition, Cambridge University Press, 1993. Cited on pages 289 and 290.

255 B.E. Schaefer, Severe limits on variations of the speed of light with frequency, *Physical Review Letters* 82, pp. 4964–4966, 21 June 1999. Cited on page 289.

256 The beginning of the modern theory of relativity is the famous paper by Albert Einstein, Zur Elektrodynamik bewegter Körper, *Annalen der Physik* 17, pp. 891–921, 1905. It still well worth reading, and every physicist should have done so. The same can be said of the famous paper, probably written after he heard of Olinto De Pretto's idea, found in Albert Einstein, Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?, *Annalen der Physik* 18, pp. 639–641, 1905. See also the review Albert Einstein, Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen, *Jahrbuch der Radioaktivität und Elektronik* 4, pp. 411–462, 1907. These papers are now available in many languages. A later, unpublished review is available in facsimile and with an English translation as Albert Einstein, Hanoch Gutfreund, ed., *Einstein's 1912 Manuscript on the Theory of Relativity*, George Braziller, 2004. Cited on pages 289, 290, and 326.

257 Albert Einstein, *Mein Weltbild*, edited by Carl Selig, Ullstein Verlag, 1998. Cited on page 289.

258 Jean van Bladel, *Relativity and Engineering*, Springer, 1984. Cited on page 289.

259 Albrecht Fölsing, *Albert Einstein – eine Biographie*, Suhrkamp p. 237, 1993. Cited on pages 289 and 299.

260 R.J. Kennedy & E.M. Thorndike, Experimental establishment of the relativity of time, *Physical Review* 42, pp. 400–418, 1932. See also H.E. Ives & G.R. Stilwell, An experimental study of the rate of a moving atomic clock, *Journal of the Optical Society of America* 28, pp. 215–226, 1938, and 31, pp. 369–374, 1941. For a modern, high-precision versions, see C. Braxmeier, H. Müller, O. Pradl, J. Mlynek, A. Peters & S. Schiller, New tests of relativity using a cryogenic optical resonator, *Physical Review Letters* 88, p. 010401, 2002. The newest result is in P. Antonini, M. Okhapkin, E. Göklü & S. Schiller, Testing the constancy of the speed of light with rotating cryogenic optical resonators, *Physical Review A* 71, p. 050101, 2005, or http://arxiv.org/abs/gr-qc/0504109. Cited on page 290.

261 Edwin F. Taylor & John A. Wheeler, *Spacetime Physics – Introduction to Special Relativity*, second edition, Freeman, 1992. See also Nick M.J. Woodhouse, *Special Relativity*, Springer, 2003. Cited on pages 290 and 334.

262 Wolfgang Rindler, *Relativity – Special, General and Cosmological*, Oxford University Press, 2001. A beautiful book by one of the masters of the field. Cited on pages 290 and 338.

263 The slowness of the speed of light inside stars is due to the frequent scattering of photons by the star matter. The most common estimate for the Sun is an escape time of 40 000 to 1 million years, but estimates between 17 000 years and 50 million years can be found in the literature. Cited on page 291.

264 L. Vestergaard Hau, S.E. Harris, Z. Dutton & C.H. Behroozi, Light speed reduction to 17 meters per second in an ultracold atomic gas, *Nature* 397, pp. 594–598, 1999. See also Ref. 26. Cited on page 291.

265 The method of explaining special relativity by drawing a few lines on paper is due to Hermann Bondi, *Relativity and Common Sense: A New Approach to Einstein*, Dover, New York, 1980. See also Dierck-Ekkehard Liebscher, *Relativitätstheorie mit Zirkel und Lineal*, Akademie-Verlag Berlin, 1991. Cited on page 291.

266 Rod S. Lakes, Experimental limits on the photon mass and cosmic vector potential, *Physical Review Letters* 80, pp. 1826–1829, 1998. The speed of light is independent of frequency within a factor of $6 \cdot 10^{-21}$, as was shown from gamma ray studies by B.E. Schaefer, Severe limits on variations of the speed of light with frequency, *Physical Review Letters* **82**, pp. 4964–4966, 1999. Cited on page 292.

267 An overview of experimental results is given in Yuan Zhong Zhang, *Special Relativity and its Experimental Foundations*, World Scientific, 1998. Cited on pages 292, 298, 307, 320, 346, and 353.

268 R.W. McGowan & D.M. Giltner, New measurement of the relativistic Doppler shift in neon, *Physical Review Letters* **70**, pp. 251–254, 1993. Cited on page 294.

269 The present record for clock synchronization seems to be 1 ps for two clocks distant 3 km from each other. See A. Valencia, G. Scarcelli & Y. Shih, Distant clock synchronization using entangled photon pairs, *Applied Physics Letters* **85**, pp. 2655–2657, 2004, or http://arxiv.org/abs/quant-ph/0407204. Cited on page 295.

270 J. Frenkel & T. Kontorowa, Über die Theorie der plastischen Verformung, *Physikalische Zeitschrift der Sowietunion* **13**, p. 1, 1938. F.C. Frank, On the equations of motion of crystal dislocations, *Proceedings of the Physical Society A* **62**, pp. 131–134, 1949. J. Eshelby, Uniformly moving dislocations, *Proceedings of the Physical Society A* **62**, pp. 307–314, 1949. See also G. Leibfried & H. Dietze, *Zeitschrift für Physik* **126**, p. 790, 1949. A general introduction can be found in A. Seeger & P. Schiller, Kinks in dislocation lines and their effects in internal friction in crystals, *Physical Acoustics* **3A**, W.P. Mason, ed., Academic Press, 1966. See also the textbooks by Frank R.N. Nabarro, *Theory of Crystal Dislocations*, Oxford University Press, 1967, or J.P. Hirth & J. Lothe, *Theory of Dislocations*, McGraw Hill, 1968. Cited on page 295.

271 This beautiful graph is taken from Z.G.T. Guiragossian, G.B. Rothbart, M.R. Yearian, R. Gearhart & J.J. Murray, Relative velocity measurements of electrons and gamma rays at 15 GeV, *Physical Review Letters* **34**, pp. 335–338, 1975. Cited on page 295.

272 To find out more about the best-known crackpots, and their ideas, send an email to majordomo@zikzak.net with the one-line body 'subscribe psychoceramics'. Cited on page 296.

273 The speed of neutrinos is the same as that of light to 9 decimal digits. This is explained by Leo Stodolsky, The speed of light and the speed of neutrinos, *Physics Letters B* **201**, p. 353, 1988. An observation of a small mass for the neutrino has been published by the Japanese Super-Kamiokande collaboration, in Y. Fukuda & al., Evidence for oscillation of atmospheric neutrinos, *Physical Review Letters* **81**, pp. 1562–1567, 1998. The newer results published by the Canadian Sudbury Neutrino Observatory, as Q.R. Ahmad & al., Direct evidence for neutrino flavor transformation from neutral-current interactions in the Sudbury Neutrino Observatory, *Physical Review Letters* **89**, p. 011301, 2002, also confirm that neutrinos have a mass in the 1 eV region. Cited on pages 297 and 1276.

274 B. Rothenstein & G. Eckstein, Lorentz transformations directly from the speed of light, *American Journal of Physics* **63**, p. 1150, 1995. See also the comment by E. Kapuścik, Comment on "Lorentz transformations directly from the speed of light," by B. Rothenstein and G. Eckstein, *American Journal of Physics* **65**, p. 1210, 1997. Cited on page 298.

275 See e.g. the 1922 lectures by Lorentz at Caltech, published as H.A. Lorentz, *Problems of Modern Physics*, edited by H. Bateman, Ginn and Company, page 99, 1927. Cited on page 298.

276 A.A. Michelson & E.W. Morley, On the relative motion of the Earth and the luminiferous ether, *American Journal of Science (3rd series)* **34**, pp. 333–345, 1887. Michelson published many other papers on the topic after this one. Cited on page 298.

277 S. Stephan, P. Antonini & M. Okhapkin, A precision test of the isotropy of the speed of light using rotating cryogenic resonators, http://arxiv.org/abs/physics/0510169. Cited on page 298.

278 H.A. Lorentz, De relative beweging van de aarde en dem aether, *Amst. Versl.* **1**, p. 74, 1892, and also H.A. Lorentz, Electromagnetic phenomena in a system moving with any velocity smaller than that of light, *Amst. Proc.* **6**, p. 809, 1904, or *Amst. Versl.* **12**, p. 986, 1904. Cited on page 302.

279 A general refutation of such proposals is discussed by S.R. Mainwaring & G.E. Stedman, Accelerated clock principles, *Physical Review A* **47**, pp. 3611–3619, 1993. Experiments on *muons* at CERN in 1968 showed that accelerations of up to 10^{20} m/s^2 have no effect, as explained by D.H. Perkins, *Introduction to High Energy Physics*, Addison-Wesley, 1972, or by J. Bailey & al., *Il Nuovo Cimento* **9A**, p. 369, 1972. Cited on page 302.

280 W. Rindler, General relativity before special relativity: an unconventional overview of relativity theory, *American Journal of Physics* **62**, pp. 887–893, 1994. Cited on page 302.

281 Steven K. Blau, Would a topology change allow Ms. Bright to travel backward in time?, *American Journal of Physics* **66**, pp. 179–185, 1998. Cited on page 305.

282 On the 'proper' formulation of relativity, see for example D. Hestenes, Proper particle mechanics, *Journal of Mathematical Physics* **15**, pp. 1768–1777, 1974. Cited on page 305.

283 The simple experiment to take a precise clock on a plane, fly it around the world and then compare it with an identical one left in place was first performed by J.C. Hafele & R.E. Keating, Around-the-world atomic clocks: predicted relativistic time gains, *Science* **177**, pp. 166–167, and Around-the-world atomic clocks: observed relativistic time gains, pp. 168–170, 14 July 1972. See also Ref. 267. Cited on page 306.

284 A readable introduction to the change of time with observers, and to relativity in general, is Roman U. Sexl & Herbert Kurt Schmidt, *Raum-Zeit-Relativität*, 2. Auflage, Vieweg & Sohn, Braunschweig, 1991. Cited on page 306.

285 Most famous is the result that moving muons stay younger, as shown for example by D.H. Frisch & J.B. Smith, Measurement of the relativistic time dilation using μ-mesons, *American Journal of Physics* **31**, pp. 342–355, 1963. For a full pedagogical treatment of the twin paradox, see E. Sheldon, Relativistic twins or sextuplets?, *European Journal of Physics* **24**, pp. 91–99, 2003. Cited on page 306.

286 Paul J. Nahin, *Time Machines – Time Travel in Physics, Metaphysics and Science Fiction*, Springer Verlag and AIP Press, second edition, 1999. Cited on page 306.

287 The first muon experiment was B. Rossi & D.B. Hall, Variation of the rate of decay of mesotrons with momentum, *Physical Review* **59**, pp. 223–228, 1941. 'Mesotron' was the old name for muon. Cited on page 307.

288 A. Harvey & E. Schucking, A small puzzle from 1905, *Physics Today*, pp. 34–36, March 2005. Cited on page 307.

289 W. Rindler, Length contraction paradox, *American Journal of Physics* **29**, pp. 365–366, 1961. For a variation without gravity, see R. Shaw, Length contraction paradox, *American Journal of Physics* **30**, p. 72, 1962. Cited on page 309.

290 H. van Lintel & C. Gruber, The rod and hole paradox re-examined, *European Journal of Physics* **26**, pp. 19–23, 2005. Cited on page 309.

291 This situation is discussed by G.P. Sastry, Is length contraction paradoxical?, *American Journal of Physics* **55**, 1987, pp. 943–946. This paper also contains an extensive literature list covering variants of length contraction paradoxes. Cited on page 309.

292 S.P. Boughn, The case of the identically accelerated twins, *American Journal of Physics* **57**, pp. 791–793, 1989. Cited on pages 310 and 313.

293 J.M. Supplee, Relativistic buoyancy, *American Journal of Physics 57* **1**, pp. 75–77, January 1989. See also G.E.A. Matsas, Relativistic Arquimedes law for fast moving bodies and the general-relativistic resolution of the 'submarine paradox', *Physical Review D* **68**, p. 027701, 2003, or http://arxiv.org/abs/gr-qc/0305106. Cited on page 310.

294 The distinction was first published by J. Terrell, Invisibility of Lorentz contraction, *Physical Review* **116**, pp. 1041–1045, 1959, and R. Penrose, The apparent shape of a relativistically moving sphere, *Proceedings of the Cambridge Philosophical Society* **55**, pp. 137–139, 1959. Cited on page 312.

295 G.R. Rybicki, Speed limit on walking, *American Journal of Physics* **59**, pp. 368–369, 1991. Cited on page 314.

296 The first examples of such astronomical observations were provided by A.R. Whitney & al., Quasars revisited: rapid time variations observed via very-long-baseline interferometry, *Science* **173**, pp. 225–230, 1971, and by M.H. Cohen & al., The small-scale structure of radio galaxies and quasi-stellar sources at 3.8 centimetres, *Astrophysical Journal* **170**, pp. 207–217, 1971. See also T.J. Pearson, S.C. Unwin, M.H. Cohen, R.P. Linfield, A.C.S. Readhead, G.A. Seielstad, R.S. Simon & R.C. Walker, Superluminal expansion of quasar 3C 273, *Nature* **290**, pp. 365–368, 1981. An overview is given in J.A. Zensus & T.J. Pearson, editors, *Superluminal radio sources*, Cambridge University Press, 1987. Another measurement, using very long baseline interferometry with radio waves, was shown on the cover of *Nature*: I.F. Mirabel & L.F. Rodriguez, A superluminal source in the galaxy, *Nature* **371**, pp. 46–48, 1994. A more recent example was reported in *Science News* **152**, p. 357, 6 December 1997.

Pedagogical explanations are given by D.C. Gabuzda, The use of quasars in teaching introductory special relativity, *American Journal of Physics* **55**, pp. 214–215, 1987, and by Edwin F. Taylor & John A. Wheeler, *Spacetime Physics – Introduction to Special Relativity*, second edition, Freeman, 1992, pages 89-92. This excellent book was mentioned already in the text. Cited on page 316.

297 O.M. Bilaniuk & E.C. Sudarshan, Particles beyond the light barrier, *Physics Today* **22**, pp. 43–51, 1969, and O.M.P. Bilanuk, V.K. Deshpande & E.C.G. Sudarshan, 'Meta' relativity, *American Journal of Physics* **30**, pp. 718–723, 1962. See also E. Recami, editor, *Tachyons, Monopoles and Related Topics*, North-Holland, Amsterdam, 1978. Cited on page 317.

298 J.P. Costella, B.H.J. McKellar, A.A. Rawlinson & G.J. Stephenson, The Thomas rotation, *American Journal of Physics* **69**, pp. 837–847, 2001. Cited on page 317.

299 See for example S.S. Costa & G.E.A. Matsas, Temperature and relativity, preprint available at http://arxiv.org/abs/gr-qc/9505045. Cited on page 318.

300 R.C. Tolman & G.N. Lewis, The principle of relativity and non-Newtonian mechanics, *Philosophical Magazine* **18**, pp. 510–523, 1909, and R.C. Tolman, Non-Newtonian mechanics: the mass of a moving body, *Philosophical Magazine* **23**, pp. 375–380, 1912. Cited on page 319.

301 S. Rainville, J.K. Thompson, E.G. Myers, J.M. Brown, M.S. Dewey, E.G. Kessler, R.D. Deslattes, H.G. Börner, M. Jentschel, P. Mutti & D.E. Pritchard, World year of physics: a direct test of $E = mc^2$, *Nature* **438**, pp. 1096–1097, 2005. Cited on page 322.

302 This information is due to a private communication by Frank DiFilippo; part of the story is given in F. DiFilippo, V. Natarajan, K.R. Boyce & D.E. Pritchard, Accurate atomic masses for fundamental metrology, *Physical Review Letters* 73, pp. 1481–1484, 1994. These measurements were performed with Penning traps; a review of the possibilities they offer is given by R.C. Thompson, Precision measurement aspects of ion traps, *Measurement Science and Technology* 1, pp. 93–105, 1990. The most important experimenters in the field of single particle levitation were awarded the Nobel Prize in 1989. One of the Nobel Prize lectures can be found in W. Paul, Electromagnetic traps for neutral and charged particles, *Reviews of Modern Physics* **62**, pp. 531–540, 1990. Cited on page 323.

303 J.L. Synge, *Relativity: The Special Theory*, North-Holland, 1956, pp. 208–213. More about antiparticles in special relativity can be found in J.P. Costella, B.H.J. McKellar & A.A. Rawlinson, Classical antiparticles, *American Journal of Physics* **65**, pp. 835–841, 1997. See also Ref. 318. Cited on page 323.

304 A. Papapetrou, Drehimpuls- und Schwerpunktsatz in der relativistischen Mechanik, *Praktika Acad. Athenes* **14**, p. 540, 1939, and A. Papapetrou, Drehimpuls- und Schwerpunktsatz in der Diracschen Theorie, *Praktika Acad. Athenes* **15**, p. 404, 1940. See also M.H.L. Pryce, The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles, *Proceedings of the Royal Society in London, A* **195**, pp. 62–81, 1948. Cited on page 325.

305 The references preceding Einstein's $E = mc^2$ are: S. Tolver Preston, *Physics of the Ether*, E. & F.N. Spon, 1875, J.H. Poincaré, La théorie de Lorentz et le principe de réaction, *Archives néerlandaises des sciences exactes et naturelles* 5, pp. 252–278, 1900, O. De Pretto, Ipotesi dell'etere nella vita dell'universo, *Reale Istituto Veneto di Scienze, Lettere ed Arti* **tomo LXIII**, parte 2, pp. 439–500, Febbraio 1904, F. Hasenöhrl, *Berichte der Wiener Akademie* **113**, p. 1039, 1904, F. Hasenöhrl, Zur Theorie der Strahlung in bewegten Körpern, *Annalen der Physik* **15**, pp. 344–370, 1904, F. Hasenöhrl, Zur Theorie der Strahlung in bewegten Körpern – Berichtigung, *Annalen der Physik* **16**, pp. 589–592, 1905. Hasenöhrl died in 1915, De Pretto in 1921. All these publications were published before the famous paper by Albert Einstein, Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?, *Annalen der Physik* **18**, pp. 639–641, 1905. Cited on page 327.

306 A jewel among the textbooks on special relativity is the booklet by Ulrich E. Schröder, *Spezielle Relativitätstheorie*, Verlag Harri Deutsch, Thun, 1981. Cited on pages 329 and 332.

307 A readable article showing a photocopy of a letter by Einstein making this point is Lev B. Okun, The concept of mass, *Physics Today*, pp. 31–36, June 1989. The topic is not without controversy, as the letters by readers following that article show; they are found in *Physics Today*, pp. 13–14 and pp. 115–117, May 1990. The topic is still a source of debates. Cited on page 331.

308 Christian Møller, *The Theory of Relativity*, Clarendon Press, 1952, 1972. This standard text has been translated in several languages. Cited on page 332.

309 P. Ehrenfest, Gleichförmige Rotation starrer Körper und Relativitätstheorie, *Physikalische Zeitschrift* **10**, pp. 918–928, 1909. Ehrenfest (incorrectly) suggested that this meant that relativity cannot be correct. A modern summary of the issue can be found in M.L. Ruggiero, The relative space: space measurements on a rotating platform, http://arxiv.org/abs/gr-qc/0309020. Cited on page 333.

310 R.J. Low, When moving clocks run fast, *European Journal of Physics* **16**, pp. 228–229, 1995. Cited on page 337.

311 G. Stephenson & C.W. Kilmister, *Special Relativity for Physicists*, Longmans, London, 1965. See also W.N. Matthews, Relativistic velocity and acceleration transformations from thought experiments, *American Journal of Physics* **73**, pp. 45–51, 2005. Cited on page 338.

312 The impossibility of defining rigid coordinate frames for non-uniformly accelerating observers is discussed by Charles Misner, Kip Thorne & John A. Wheeler, *Gravitation*, Freeman, p. 168, 1973. Cited on page 340.

313 E.A. Desloge & R.J. Philpott, Uniformly accelerated reference frames in special relativity, *American Journal of Physics* 55, pp. 252–261, 1987. Cited on page 340.

314 R.H. Good, Uniformly accelerated reference frame and twin paradox, *American Journal of Physics* **50**, pp. 232–238, 1982. Cited on pages 340, 341, and 344.

315 J. Dwayne Hamilton, The uniformly accelerated reference frame, *American Journal of Physics* **46**, pp. 83–89, 1978. Cited on page 341.

316 The best and cheapest mathematical formula collection remains the one by K. Rottmann, *Mathematische Formelsammlung*, BI Hochschultaschenbücher, 1960. Cited on page 341.

317 C.G. Adler & R.W. Brehme, Relativistic solutions to a falling body in a uniform gravitation field, *American Journal of Physics* **59**, pp. 209–213, 1991. Cited on page 342.

318 See for example the excellent lecture notes by D.J. Raymond, A radically modern approach to freshman physics, on the http://www.physics.nmt.edu/~raymond/teaching.html website. Cited on pages 342 and 355.

319 L. Mishra, The relativistic acceleration addition theorem, *Classical and Quantum Gravity* **11**, pp. L97–L102, 1994. Cited on page 342.

320 Edward A. Desloge, The gravitational red-shift in a uniform field, *American Journal of Physics* 58, pp. 856–858, 1990. Cited on page 344.

321 One of the latest of these debatable experiments is T.P. Krisher, L. Maleki, G.F. Lutes, L.E. Primas, R.T. Logan, J.D. Anderson & C.M. Will, Test of the isotropy of the one-way speed of light using hydrogen-maser frequency standards, *Physical Review D* **42**, pp. 731–734, 1990. Cited on page 346.

322 Edwin F. Taylor & A.P. French, Limitation on proper length in special relativity, *American Journal of Physics* **51**, pp. 889–893, 1983. Cited on page 347.