THE SPECIAL THEORY OF RELATIVITY

Lecture Notes prepared by

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1 Introduction: What is Relativity?

Until the end of the 19th century it was believed that Newton's three Laws of Motion and the associated ideas about the properties of space and time provided a basis on which the motion of matter could be completely understood. However, the formulation by Maxwell of a unified theory of electromagnetism disrupted this comfortable state of affairs – the theory was extraordinarily successful, yet at a fundamental level it seemed to be inconsistent with certain aspects of the Newtonian ideas of space and time. Ultimately, a radical modification of these latter concepts, and consequently of Newton's equations themselves, was found to be necessary. It was Albert Einstein who, by combining the experimental results and physical arguments of others with his own unique insights, first formulated the new principles in terms of which space, time, matter and energy were to be understood. These principles, and their consequences constitute the Special Theory of Relativity. Later, Einstein was able to further develop this theory, leading to what is known as the General Theory of Relativity. Amongst other things, this latter theory is essentially a theory of gravitation. The General Theory will not be dealt with in this course.

Relativity (both the Special and General) theories, quantum mechanics, and thermodynamics are the three major theories on which modern physics is based. What is unique about these three theories, as distinct from say the theory of electromagnetism, is their generality. Embodied in these theories are general principles which all more specialized or more specific theories are required to satisfy. Consequently these theories lead to general conclusions which apply to all physical systems, and hence are of enormous power, as well as of fundamental significance. The role of relativity appears to be that of specifying the properties of space and time, the arena in which all physical processes take place.

It is perhaps a little unfortunate that the word 'relativity' immediately conjures up thoughts about the work of Einstein. The idea that a principle of relativity applies to the properties of the physical world is very old: it certainly predates Newton and Galileo, but probably not as far back as Aristotle. What the principle of relativity essentially states is the following:

> *The laws of physics take the same form in all frames of reference moving with constant velocity with respect to one another.*

This is a statement that can be given a precise mathematical meaning: the laws of physics are expressed in terms of equations, and the form that these equations take in different reference frames moving with constant velocity with respect to one another can be calculated by use of transformation equations – the so-called Galilean transformation in the case of Newtonian relativity. The principle of relativity then requires that the transformed equations have exactly the same form in all frames of reference, in other words that the physical laws are the same in all frames of reference.

This statement contains concepts which we have not developed, so perhaps it is best at this stage to illustrate its content by a couple of examples. First consider an example from 'everyday experience' – a train carriage moving smoothly at a constant speed on a straight and level track – this is a 'frame of reference'. Suppose that in this carriage is a pool table. If you were a passenger on this carriage and you decided to play a game of pool, one of the first things that you would notice is that in playing any shot, you would have to make no allowance whatsoever for the motion of the train. Any judgement of how to play a shot as learned by playing the game back home, or in the local pool hall,

would apply equally well on the train, irrespective of how fast the train was moving. If we consider that what is taking place here is the innate application of Newton's Laws to describe the motion and collision of the pool balls, we see that no adjustment has to be made to these laws when playing the game on the moving train.

This argument can be turned around. Suppose the train windows are covered, and the carriage is well insulated so that there is no immediate evidence to the senses as to whether or not the train is in motion. It might nevertheless still be possible to determine if the train is in motion by carrying out an experiment, such as playing a game of pool. But, as described above, a game of pool proceeds in exactly the same way as if it were being played back home – no change in shot-making is required. There is no indication from this experiment as to whether or not the train is in motion. There is no way of knowing whether, on pulling back the curtains, you are likely to see the countryside hurtling by, or to find the train sitting at a station. In other words, what the principle of relativity means is that it is not possible to determine whether or not the train carriage is moving.

This idea can be extended to encompass other laws of physics. To this end, imagine a collection of spaceships with engines shut off all drifting through space. Each space ship constitutes a 'frame of reference', an idea that will be better defined later. On each of these ships a series of experiments is performed: a measurement of the half life of uranium, a measurement of the outcome of the collision of two billiard balls, an experiment in thermodynamics, e.g. a measurement of the specific heat of a substance, a measurement of the speed of light radiated from a nearby star: any conceivable experiment. If the results of these experiments are later compared, what is found is that in all cases (within experimental error) the results are identical. In other words, the various laws of physics being tested here yield exactly the same results for all the spaceships, in accordance with the principle of relativity.

Thus, quite generally the principle of relativity means that it is not possible, by considering any physical process whatsoever, to determine whether or not one or the other of the spaceships is 'in motion'. The results of all the experiments are the same on all the space ships, so there is nothing that definitely singles out one space ship over any other as being the one that is stationary. It is true that from the point of view of an observer on any one of the space ships that it is the others that are in motion. But the same statement can be made by an observer in *any* space ship. All that we can say for certain is that the space ships are in relative motion, and not claim that one of them is 'truly' stationary, while the others are all 'truly' moving.

This principle of relativity was accepted (in somewhat simpler form i.e. with respect to the mechanical behaviour of bodies) by Newton and his successors, even though Newton postulated that underlying it all was 'absolute space' which defined the state of absolute rest. He introduced the notion in order to cope with the difficulty of specifying with respect to what an accelerated object is being accelerated. To see what is being implied here, imagine space completely empty of all matter except for two masses joined by a spring. Now suppose that the arrangement is rotated, that is, they undergo acceleration. Naively, in accordance with our experience, we would expect that the masses would pull apart. But why should they? How do the masses 'know' that they are being rotated? There are no 'signposts' in an otherwise empty universe that would indicate that rotation is taking place. By proposing that there existed an absolute space, Newton was able to claim that the masses are being accelerated with respect to this absolute space, and hence that they would separate in the way expected for masses in circular motion. But this was a supposition made more for the convenience it offered in putting together his Laws of motion, than anything else. It was an assumption that could not be substantiated, as

Newton was well aware – he certainly felt misgivings about the concept! Other scientists were more accepting of the idea, however, with Maxwell's theory of electromagnetism for a time seeming to provide some sort of confirmation of the concept.

One of the predictions of Maxwell's theory was that light was an electromagnetic wave that travelled with a speed $c \approx 3 \times 10^8$ ms^{-1}. But relative to what? Maxwell's theory did not specify any particular frame of reference for which light would have this speed. A convenient resolution to this problem was provided by an already existing assumption concerning the way light propagated through space. That light was a form of wave motion was well known – Young's interference experiments had shown this – but the Newtonian world view required that a wave could not propagate through empty space: there must be present a medium of some sort that vibrated as the waves passed, much as a string vibrates as a wave travels along it. The proposal was therefore made that space was filled with a substance known as the ether whose purpose was to be the medium that vibrated as the light waves propagated through it. It was but a small step to then propose that this ether was stationary with respect to Newton's absolute space, thereby solving the problem of what the frame of reference was in which light had the speed c. Furthermore, in keeping with the usual ideas of relative motion, the thinking then was then that if you were to travel relative to the ether towards a beam of light, you would measure its speed to be greater than c, and less than c if you travelled away from the beam. It then came as an enormous surprise when it was found experimentally that this was not, in fact, the case.

This discovery was made by Michelson and Morley, who fully accepted the ether theory, and who, quite reasonably, thought it would be a nice idea to try to measure how fast the earth was moving through the ether. They found to their enormous surprise that the result was always zero irrespective of the position of the earth in its orbit around the sun or, to put it another way, they measured the speed of light always to be the same value c whether the light beam was moving in the same direction or the opposite direction to the motion of the earth in its orbit. In our spaceship picture, this is equivalent to all the spaceships obtaining the same value for the speed of light radiated by the nearby star irrespective of their motion relative to the star. This result is completely in conflict with the rule for relative velocities, which in turn is based on the principle of relativity as enunciated by Newton and Galileo. Thus the independence of the speed of light on the motion of the observer seems to take on the form of an immutable law of nature, and yet it is apparently inconsistent with the principle of relativity. Something was seriously amiss, and it was Einstein who showed how to get around the problem, and in doing so he was forced to conclude that space and time had properties undreamt of in the Newtonian world picture.

All these ideas, and a lot more besides, have to be presented in a much more rigorous form. The independence of results of the hypothetical experiments described above on the state of motion of the experimenters can be understood at a fundamental level in terms of the mathematical forms taken by the laws of nature. All laws of nature appear to have expression in mathematical form, so what the principle of relativity can be understood as saying is that the equations describing a law of nature take the same mathematical form in all inertial frames of reference. It is this latter perspective on relativity that is developed here, and an important starting point is the notion of a frame of reference.

2 Frames of Reference

Newton's laws are, of course, the laws which determine how matter moves through space as a function of time. So, in order to give these laws a precise meaning we have to specify how we measure the position of some material object, a particle say, and the time at which it is at that position. We do this by introducing the notion of a frame of reference.

2.1 A Framework of Rulers and Clocks

First of all we can specify the positions of the particle in space by determining its coordinates relative to a set of mutually perpendicular axes X, Y, Z. In practice this could be done by choosing our origin of coordinates to be some convenient point and imagining that rigid rulers – which we can also imagine to be as long as necessary – are laid out from this origin along these three mutually perpendicular directions. The position of the particle can then be read off from these rulers, thereby giving the three position coordinates (x, y, z) of the particle[1].

We are free to set up our collection of rulers anywhere we like e.g. the origin could be some fixed point on the surface of the earth and the rulers could be arranged to measure x and y positions horizontally, and z position vertically. Alternatively we could imagine that the origin is a point on a rocket travelling through space, or that it coincides with the position of a subatomic particle, with the associated rulers being carried along with the moving rocket or particle[2].

By this means we can specify *where* the particle is. In order to specify *when* it is at a particular point in space we can stretch our imagination further and imagine that in addition to having rulers to measure position, we also have at each point in space a clock, and that these clocks have all been *synchronized* in some way. The idea is that with these clocks we can tell when a particle is at a particular position in space simply by reading off the time indicated by the clock at that position.

According to our 'common sense' notion of time, it would appear sufficient to have only one set of clocks filling all of space. Thus, no matter which set of moving rulers we use to specify the position of a particle, we always use the clocks belonging to this single vast set to tell us when a particle is at a particular position. In other words, there is only one 'time' for all the position measuring set of rulers. This time is the same time independent of how the rulers are moving through space. This is the idea of universal or absolute time due to Newton. However, as Einstein was first to point out, this idea of absolute time is untenable, and that the measurement of time intervals (e.g. the time interval between two events such as two supernovae occurring at different positions in space) will in fact differ for observers in motion relative to each other. In order to prepare ourselves for this possibility, we shall suppose that *for each possible set of rulers* – including those fixed relative to the ground, or those moving with a subatomic particle and so on, there are a *different* set of clocks. Thus the position measuring rulers carry their own set of clocks around with them. The clocks belonging to each set of rulers are of course synchronized with respect to each other. Later on we shall see how this synchronization can be achieved.

[1] Probably a better construction is to suppose that space is filled with a scaffolding of rods arranged in a three dimensional grid.

[2] If, for instance, the rocket has its engines turned on, we would be dealing with an accelerated frame of reference in which case more care is required in defining how position (and time) can be measured in such a frame. Since we will ultimately be concerning ourselves with non-accelerated observers, we will not concern ourselves with these problems. A proper analysis belongs to General Relativity.

The idea now is that relative to a particular set of rulers we are able to specify where a particle is, and by looking at the clock (belonging to that set of rulers) at the position of the particle, we can specify when the particle is at that position. Each possible collection of rulers and associated clocks constitutes what is known as a frame of reference or a reference frame.

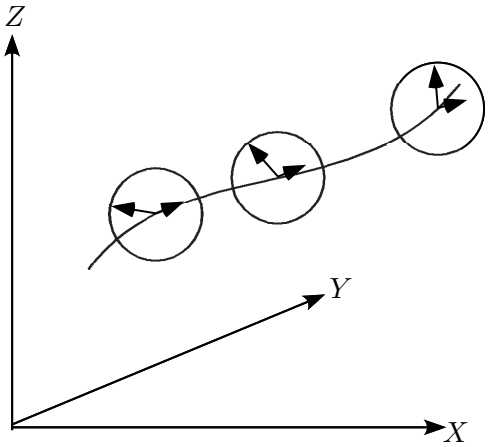

Figure 1: Path of a particle as measured in a frame of reference. The clocks indicate the times at which the particle passed the various points along the way.

In many texts reference is often made to an observer in a frame of reference whose job apparently is to make various time and space measurements within this frame of reference. Unfortunately, this conjures up images of a person armed with a stopwatch and a pair of binoculars sitting at the origin of coordinates and peering out into space watching particles (or planets) collide, stars explode and so on. This is not the sense in which the term observer is to be interpreted. It is important to realise that measurements of time are made using clocks which are positioned at the spatial point at which an event occurs. Any centrally positioned observer would have to take account of the time of flight of a signal to his or her observation point in order to calculate the actual time of occurrence of the event. One of the reasons for introducing this imaginary ocean of clocks is to avoid such a complication. Whenever the term observer arises it should be interpreted as meaning the reference frame itself, except in instances in which it is explicitly the case that the observations of an isolated individual are under consideration.

If, as measured by one particular set of rulers and clocks (i.e. frame of reference) a particle is observed to be at a position at a time t (as indicated by the clock at (x, y, z)), we can summarize this information by saying that the particle was observed to be at the point (x, y, z, t) in space-time. The motion of the particle relative to this frame of reference would be reflected in the particle being at different positions (x, y, z) at different times t. For instance in the simplest non-trivial case we may find that the particle is moving at constant speed v in the direction of the positive X axis, i.e. $x = vt$. However, if the motion of the same particle is measured relative to a frame of reference attached to say a butterfly fluttering erratically through the air, the positions (x', y', z') at different times t' (given by a series of space time points) would indicate the particle moving on an erratic path relative to this new frame of reference. Finally, we could consider the frame of reference whose spatial origin coincides with the particle itself. In this last case, the position of the particle *does not change* since it remains at the spatial origin of its frame of reference. However, the clock associated with this origin keeps on ticking so that the particle's coordinates in space-time are $(0, 0, 0, t)$ with t the time indicated on the clock at the origin, being the only quantity that changes. If a particle remains stationary relative

to a particular frame of reference, then that frame of reference is known as the *rest frame* for the particle.

Of course we can use frames of reference to specify the where and when of things other than the position of a particle at a certain time. For instance, the point in space-time at which an explosion occurs, or where and when two particles collide etc., can also be specified by the four numbers (x, y, z, t) relative to a particular frame of reference. In fact any event occurring in space and time can be specified by four such numbers whether it is an explosion, a collision or the passage of a particle through the position (x, y, z) at the time t. For this reason, the four numbers (x, y, z, t) together are often referred to as an *event*.

2.2 Inertial Frames of Reference and Newton's First Law of Motion

Having established how we are going to measure the coordinates of a particle in space and time, we can now turn to considering how we can use these ideas to make a statement about the physical properties of space and time. To this end let us suppose that we have somehow placed a particle in the depths of space far removed from all other matter. It is reasonable to suppose that a particle so placed is *acted on by no forces whatsoever*[3]. The question then arises: 'What kind of motion is this particle undergoing?' In order to determine this we have to measure its position as a function of time, and to do this we have to provide a reference frame. We could imagine all sorts of reference frames, for instance one attached to a rocket travelling in some complicated path. Under such circumstances, the path of the particle as measured relative to such a reference frame would be very complex. However, it is at this point that an assertion can be made, namely that for certain frames of reference, the particle will be travelling in a particularly simple fashion – a straight line at constant speed. This is something that has not and possibly could not be confirmed experimentally, but it is nevertheless accepted as a true statement about the properties of the motion of particles in the absences of forces. In other words we can adopt as a law of nature, the following statement:

> *There exist frames of reference relative to which a particle acted on by no forces moves in a straight line at constant speed.*

This essentially a claim that we are making about the properties of spacetime. It is also simply a statement of Newton's First Law of Motion. A frame of reference which has this property is called an inertial frame of reference, or just an inertial frame.

Gravity is a peculiar force in that if a reference frame is freely falling under the effects of gravity, then any particle also freely falling will be observed to be moving in a straight line at constant speed relative to this freely falling frame. Thus freely falling frames constitute inertial frames of reference, at least locally.

3 The Galilean Transformation

The above argument does not tell us whether there is one or many inertial frames of reference, nor, if there is more than one, does it tell us how we are to relate the coordinates

[3]It is not necessary to define what we mean by force at this point. It is sufficient to presume that if the particle is far removed from all other matter, then its behaviour will in no way be influenced by other matter, and will instead be in response to any inherent properties of space (and time) in its vicinity.

of an event as observed from the point-of-view of one inertial reference frame to the coordinates of the same event as observed in some other. In establishing the latter, we can show that there is in fact an infinite number of inertial reference frames. Moreover, the transformation equations that we derive are then the mathematical basis on which it can be shown that Newton's Laws are consistent with the principle of relativity. To derive these transformation equations, consider an inertial frame of reference S and a second reference frame S' moving with a velocity v_x relative to S.

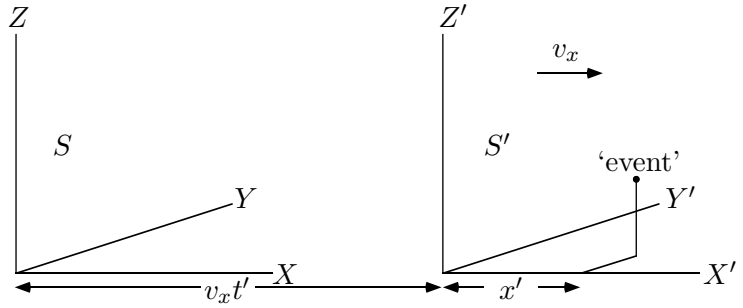

Figure 2: A frame of reference S' is moving with a velocity v_x relative to the inertial frame S. An event occurs with spatial coordinates (x, y, z) at time t in S and at (x', y', z') at time t' in S'.

Let us suppose that the clocks in S and S' are set such that when the origins of the two reference frames O and O' coincide, all the clocks in both frames of reference read zero i.e. $t = t' = 0$. According to 'common sense', if the clocks in S and S' are synchronized at $t = t' = 0$, then they will always read the same, i.e. $t = t'$ always. This, once again, is the absolute time concept introduced in Section 2.1. Suppose now that an event of some kind, e.g. an explosion, occurs at a point (x', y', z', t') according to S'. Then, by examining Fig. (2), according to S, it occurs at the point

$$x = x' + v_x t', \quad y = y', \quad z = z'$$

and at the time $$t = t' \tag{1}$$

These equations together are known as the Galilean Transformation, and they tell us how the coordinates of an event in one inertial frame S are related to the coordinates of the same event as measured in another frame S' moving with a constant velocity relative to S.

Now suppose that in inertial frame S, a particle is acted on by no forces and hence is moving along the straight line path given by:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t \tag{2}$$

where $\mathbf{u}$ is the velocity of the particle as measured in S. Then in S', a frame of reference moving with a velocity $\mathbf{v} = v_x\mathbf{i}$ relative to S, the particle will be following a path

$$\mathbf{r}' = \mathbf{r}_0 + (\mathbf{u} - \mathbf{v})t' \tag{3}$$

where we have simply substituted for the components of $\mathbf{r}$ using Eq. (1) above. This last result also obviously represents the particle moving in a straight line path at constant speed. And since the particle is being acted on by no forces, S' is also an inertial frame, and since $\mathbf{v}$ is arbitrary, there is in general an infinite number of such frames.

Incidentally, if we take the derivative of Eq. (3) with respect to t, and use the fact that $t = t'$, we obtain

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \tag{4}$$

which is the familiar addition law for relative velocities.

It is a good exercise to see how the inverse transformation can be obtained from the above equations. We can do this in two ways. One way is simply to solve these equations so as to express the primed variables in terms of the unprimed variables. An alternate method, one that is mpre revealing of the underlying symmetry of space, is to note that if S' is moving with a velocity v_x with respect to S, then S will be moving with a velocity $-v_x$ with respect to S' so the inverse transformation should be obtainable by simply exchanging the primed and unprimed variables, and replacing v_x by $-v_x$. Either way, the result obtained is

$$\left.\begin{aligned} x' &= x - v_x t \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned}\right\} \tag{5}$$

4 Newtonian Force and Momentum

Having proposed the existence of a special class of reference frames, the inertial frames of reference, and the Galilean transformation that relates the coordinates of events in such frames, we can now proceed further and study whether or not Newton's remaining laws of motion are indeed consistent with the principle of relativity. FIrst we need a statement of these two further laws of motion.

4.1 Newton's Second Law of Motion

It is clearly the case that particles do not always move in straight lines at constant speeds relative to an inertial frame. In other words, a particle can undergo acceleration. This deviation from uniform motion by the particle is attributed to the action of a force. If the particle is measured in the inertial frame to undergo an acceleration $\mathbf{a}$, then this acceleration is a consequence of the action of a force $\mathbf{F}$ where

$$\mathbf{F} = m\mathbf{a} \tag{6}$$

and where the mass m is a constant characteristic of the particle and is assumed, in Newtonian dynamics, to be the same in all inertial frames of reference. This is, of course, a statement of Newton's Second Law. This equation relates the force, mass and acceleration of a body as measured relative to a particular inertial frame of reference.

As we indicated in the previous section, there are in fact an infinite number of inertial frames of reference and it is of considerable importance to understand what happens to Newton's Second Law if we measure the force, mass and acceleration of a particle from different inertial frames of reference. In order to do this, we must make use of the Galilean transformation to relate the coordinates (x, y, z, t) of a particle in one inertial frame S say to its coordinates (x', y', z', t') in some other inertial frame S'. But before we do this, we also need to look at Newton's Third Law of Motion.

4.2 Newton's Third Law of Motion

Newton's Third Law, namely that to every action there is an equal and opposite reaction, can also be shown to take the same form in all inertial reference frames. This is not done directly as the statement of the Law just given is not the most useful way that it can be presented. A more useful (and in fact far deeper result) follows if we combine the Second and Third Laws, leading to the law of conservation of momentum which is

> *In the absence of any external forces, the total momentum of a system is constant.*

It is then a simple task to show that if the momentum is conserved in one inertial frame of reference, then via the Galilean transformation, it is conserved in all inertial frames of reference.

5 Newtonian Relativity

By means of the Galilean Transformation, we can obtain an important result of Newtonian mechanics which carries over in a much more general form to special relativity. We shall illustrate the idea by means of an example involving two particles connected by a spring. If the X coordinates of the two particles are x_1 and x_2 relative to some reference frame S then from Newton's Second Law the equation of motion of the particle at $x_1$is

$$m_1 \frac{d^2 x_1}{dt^2} = -k(x_1 - x_2 - l) \tag{7}$$

where k is the spring constant, l the natural length of the spring, and m_1 the mass of the particle. If we now consider the same pair of masses from the point of view of another frame of reference S' moving with a velocity v_x relative to S, then

$$x_1 = x_1' + v_x t' \quad \text{and} \quad x_2 = x_2' + v_x t' \tag{8}$$

so that

$$\frac{d^2 x_1}{dt^2} = \frac{d^2 x_1'}{dt'^2} \tag{9}$$

and

$$x_2 - x_1 = x_2' - x_1'. \tag{10}$$

Thus, substituting the last two results into Eq. (7) gives

$$m_1 \frac{d^2 x_1'}{dt'^2} = -k(x_1' - x_2' - l) \tag{11}$$

Now according to Newtonian mechanics, the mass of the particle is the same in both frames i.e.

$$m_1 = m_1' \tag{12}$$

where m_1' is the mass of the particle as measured in S'. Hence

$$m_1' \frac{d^2 x_1'}{dt'^2} = -k(x_1' - x_2' - l) \tag{13}$$

which is exactly the same equation as obtained in S, Eq. (7) except that the variables x_1 and x_2 are replaced by x_1' and x_2'. In other words, the *form* of the equation of motion

derived from Newton's Second Law is the same in both frames of reference. This result can be proved in a more general way than for than just masses on springs, and we are lead to conclude that the mathematical form of the equations of motion obtained from Newton's Second Law are the same in all inertial frames of reference.

Continuing with this example, we can also show that momentum is conserved in all inertial reference frames. Thus, in reference frame S, the total momentum is

$$m_1\dot{x}_1 + m_2\dot{x}_2 = P = \text{ constant.} \tag{14}$$

Using Eq. (8) above we then see that in S' the total momentum is

$$P' = m_1'\dot{x}_1' + m_2'\dot{x}_2' = m_1\dot{x}_1 + m_2\dot{x}_2 - (m_1 + m_2)v_x = P - (m_1 + m_2)v_x \tag{15}$$

which is also a constant (but not the same constant as in S – it is not required to be the same constant!!). The analogous result to this in special relativity plays a very central role in setting up the description of the dynamics of a system.

The general conclusion we can draw from all this is that:

Newton's Laws of motion are identical in all inertial frames of reference.

This is the Newtonian (or Galilean) principle of relativity, and was essentially accepted by all physicists, at least until the time when Maxwell put together his famous set of equations. One consequence of this conclusion is that it is not possible to determine whether or not a frame of reference is in a state of motion by any experiment involving Newton's Laws. At no stage do the Laws depend on the velocity of a frame of reference relative to anything else, even though Newton had postulated the existence of some kind of "absolute space" i.e. a frame of reference which defined the state of absolute rest, and with respect to which the motion of anything could be measured. The existence of such a reference frame was taken for granted by most physicists, and for a while it was thought to be have been uncovered following on from the appearance on the scene of Maxwell's theory of electromagnetism.

6 Maxwell's Equations and the Ether

The Newtonian principle of relativity had a successful career till the advent of Maxwell's work in which he formulated a mathematical theory of electromagnetism which, amongst other things, provided a successful physical theory of light. Not unexpectedly, it was anticipated that the equations Maxwell derived should also obey the above Newtonian principle of relativity in the sense that Maxwell's equations should also be the same in all inertial frames of reference. Unfortunately, it was found that this was not the case. Maxwell's equations were found to assume completely different forms in different inertial frames of reference. It was as if $\mathbf{F} = m\mathbf{a}$ worked in one frame of reference, but in another, the law had to be replaced by some bizarre equation like $\mathbf{F}' = m(\mathbf{a}')^2\mathbf{a}'$! In other words it appeared as if Maxwell's equations took a particularly simple form in one special frame of reference, but a quite complicated form in another moving relative to this special reference frame. For instance, the wave equation for light assumed the simple form

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{16}$$

in this 'special frame' S, which is the equation for waves moving at the speed c. Under the Galilean transformation, this equation becomes

$$\frac{\partial^2 E'}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E'}{\partial t'^2} - \frac{2v_x}{c^2}\frac{\partial^2 E'}{\partial x' \partial t'} - \frac{v_x}{c^2}\frac{\partial}{\partial x'}\left[v_x \frac{\partial E'}{\partial x'}\right] = 0 \tag{17}$$

for a frame S' moving with velocity v_x relative to S. This 'special frame' S was assumed to be the one that defined the state of absolute rest as postulated by Newton, and that stationary relative to it was a most unusual entity, the ether. The ether was a substance that was supposedly the medium in which light waves were transmitted in a way something like the way in which air carries sound waves. Consequently it was believed that the behaviour of light, in particular its velocity, as measured from a frame of reference moving relative to the ether would be different from its behaviour as measured from a frame of reference stationary with respect to the ether. Since the earth is following a roughly circular orbit around the sun, then it follows that a frame of reference attached to the earth must at some stage in its orbit be moving relative to the ether, and hence a change in the velocity of light should be observable at some time during the year. From this, it should be possible to determine the velocity of the earth relative to the ether. An attempt was made to measure this velocity. This was the famous experiment of Michelson and Morley. Simply stated, they argued that if light is moving with a velocity c through the ether, and the earth was moving with a velocity v relative to the ether, then light should be observed to be travelling with a velocity $c' = c - v$ at some stage in the Earth's orbit relative to the Earth. We can see this by simply solving the wave equation in S:

$$E(x,t) = E(x - ct) \tag{18}$$

where we are supposing that the wave is travelling in the positive X direction. If we now apply the Galilean Transformation to this expression, we get, for the field $E'(x',t')$ as measured in S', the result

$$E'(x',t') = E(x,t) = E(x' + v_x t' - ct') = E(x' - (c - v_x)t') \tag{19}$$

i.e. the wave is moving with a speed $c - v_x$ which is just the Galilean Law for the addition of velocities given in Eq. (4).

Needless to say, on performing their experiment – which was extremely accurate – they found that the speed of light was always the same. Obviously something was seriously wrong. Their experiments seemed to say that the earth was not moving relative to the ether, which was manifestly wrong since the earth was moving in a circular path around the sun, so at some stage it had to be moving relative to the ether. Many attempts were made to patch things up while still retaining the same Newtonian ideas of space and time. Amongst other things, it was suggested that the earth dragged the ether in its immediate vicinity along with it. It was also proposed that objects contracted in length along the direction parallel to the direction of motion of the object relative to the ether. This suggestion, due to Fitzgerald and elaborated on by Lorentz and hence known as the Lorentz-Fitzgerald contraction, 'explained' the negative results of the Michelson-Morley experiment, but faltered in part because no physical mechanism could be discerned that would be responsible for the contraction. The Lorentz-Fitzgerald contraction was to resurface with a new interpretation following from the work of Einstein. Thus some momentary successes were achieved, but eventually all these attempts were found to be unsatisfactory in various ways. It was Einstein who pointed the way out of the impasse, a way out that required a massive revision of our concepts of space, and more particularly, of time.

7 Einstein's Postulates

The difficulty that had to be resolved amounted to choosing amongst three alternatives:

1. The Galilean transformation was correct and something was wrong with Maxwell's equations.
2. The Galilean transformation applied to Newtonian mechanics only.
3. The Galilean transformation, and the Newtonian principle of relativity based on this transformation were wrong and that there existed a new relativity principle valid for both mechanics and electromagnetism that was not based on the Galilean transformation.

The first possibility was thrown out as Maxwell's equations proved to be totally successful in application. The second was unacceptable as it seemed something as fundamental as the transformation between inertial frames could not be restricted to but one set of natural phenomena i.e. it seemed preferable to believe that physics was a unified subject. The third was all that was left, so Einstein set about trying to uncover a new principle of relativity. His investigations led him to make two postulates:

1. All the laws of physics are the same in every inertial frame of reference. This postulate implies that there is no experiment whether based on the laws of mechanics or the laws of electromagnetism from which it is possible to determine whether or not a frame of reference is in a state of uniform motion.
2. The speed of light is independent of the motion of its source.

Einstein was inspired to make these postulates[4] through his study of the properties of Maxwell's equations and not by the negative results of the Michelson-Morley experiment, of which he was apparently only vaguely aware. It is this postulate that forces us to reconsider what we understand by space and time.

One immediate consequence of these two postulates is that the speed of light is the same in all inertial frames of reference. We can see this by considering a source of light and two frames of reference, the first frame of reference S' stationary relative to the source of light and the other, S, moving relative to the source of light.

[4] Einstein also tacitly made three further assumptions

Homogeneity: The intrinsic properties of empty space are the same everywhere and for all time. In other words, the properties of the rulers and clocks do not depend on their positions in (empty) space, nor do they vary over time.

Spatial Isotropy: The intrinsic properties of space is the same in all directions. In other words, the properties of the rulers and clocks do not depend on their orientations in empty space.

No Memory: The extrinsic properties of the rulers and clocks may be functions of their current states of motion, but not of their previous states of motion.

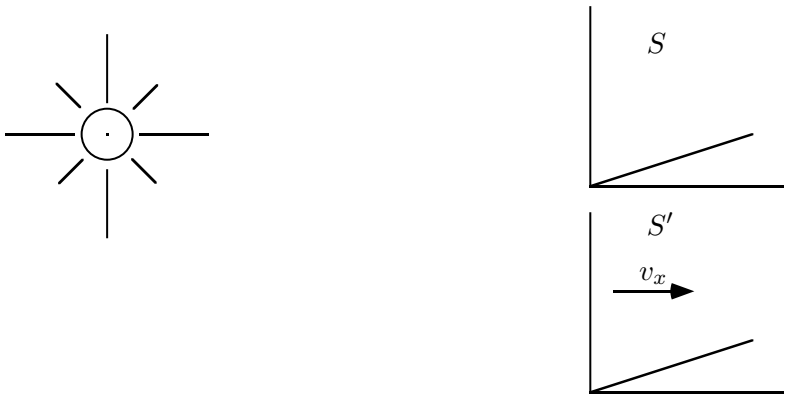

Figure 3: A source of light observed from two inertial frames S and S' where S' is moving with a velocity v_x with respect to S.

By postulate 2, S measures the speed of light to be c. However, from postulate 1, this situation is indistinguishable from that depicted in Fig. (4)

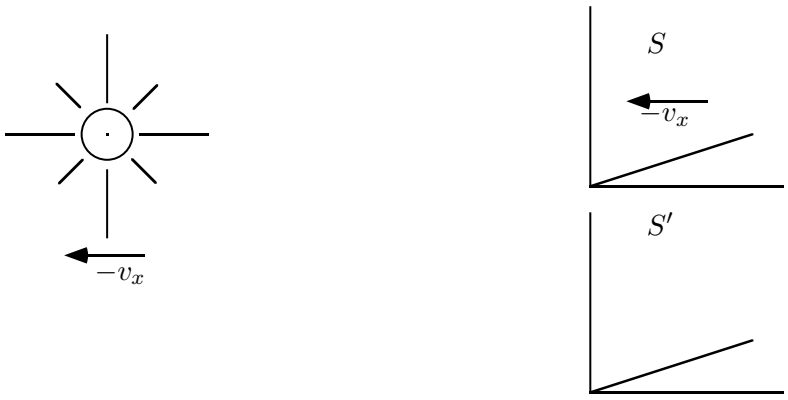

Figure 4: The same situation as in Fig. (3) except from the point of view of S'.

and by postulate 2, must also measure the speed of light to be c. In other words, both reference frames S and measure the speed of light to be c.

Before proceeding further with the consequences of these two rather innocent looking postulates, we have to be more precise about how we go about measuring time in an inertial frame of reference.

8 Clock Synchronization in an Inertial Frame

Recall from Section 2.1 that in order to measure the time at which an event occurred at a point in space, we assumed that all of space was filled with clocks, one for each point in space. Moreover, there were a separate set of clocks for each set of rulers so that a frame of reference was defined both by these rulers and by the set of clocks which were carried along by the rulers. It was also stated that all the clocks in each frame of reference were synchronized in some way, left unspecified. At this juncture it is necessary to be somewhat more precise about how this synchronization is to be achieved. The necessity for doing this lies in the fact that we have to be very clear about what we are doing when we are comparing the times of occurrence of events, particularly when the events occur at two spatially separate points.

The procedure that can be followed to achieve the synchronization of the clocks in one frame of reference is quite straightforward. We make use of the fact that the speed of light is precisely known, and is assumed to be always a constant everywhere in free space no matter how it is generated or in which direction it propagates through space. The synchronization is then achieved in the following way. Imagine that at the spatial origin of the frame of reference we have a master clock, and that at some instant $t_0 = 0$ indicated by this clock a spherical flash of light is emitted from the source.

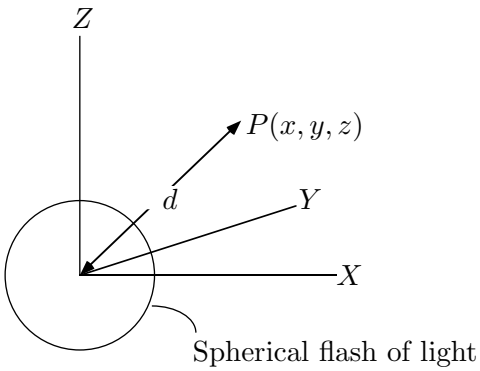

Figure 5: A spherical flash of light emitted at $t = 0$ propagates out from the origin, reaching the point P after a time d/c. The clock at P is then set to read $t = d/c$.

The flash of light will eventually reach the point $P(x, y, z)$ situated a distance d from the origin O. When this flash reaches P, the clock at that position is adjusted to read $t = d/c$. And since $d^2 = x^2 + y^2 + z^2$, this means that

$$x^2 + y^2 + z^2 = c^2t^2 \tag{20}$$

a result made use of later in the derivation of the Lorentz equations.

This procedure is followed for all the clocks throughout the frame of reference. By this means, the clocks can be synchronized. A similar procedure applies for every frame of reference with its associated clocks.

It should be pointed out that it is not necessary to use light to do this. We could have used any collection of objects whose speed we know with great precision. However it is a reasonable choice to use light since all evidence indicates that light always travels with the same speed c everywhere in space. Moreover, when it comes to comparing observations made in different frames of reference, we can exploit the fact the speed of light always has the same value through postulate 2 above. We do not know as yet what happens for any other objects. In fact, as a consequence of Einstein's second postulate we find that whereas the clocks in one reference frame have all been synchronized to everyone's satisfaction in that frame of reference, it turns out that they are not synchronized with respect to another frame of reference moving with respect to the first. The meaning and significance of this lack of synchronization will be discussed later.

We are now in a position to begin to investigate how the coordinates of an event as measured in one frame of reference are related to the coordinates of the event in another frame of reference. This relationship between the two sets of coordinates constitutes the so-called Lorentz transformation.

9 Lorentz Transformation

In deriving this transformation, we will eventually make use of the constancy of the speed of light, but first we will derive the general form that the transformation law must take purely from kinematic/symmetry considerations. The starting point is to consider two inertial frames S and S' where S' is moving with a velocity v_x relative to S.

Let us suppose that when the two origins coincide, the times on the clocks in each frame of reference are set to read zero, that is $t = t' = 0$. Now consider an event that occurs at the point (x, y, z, t) as measured in S. The same event occurs at (x', y', z', t') in S'. What we are after is a set of equations that relate these two sets of coordinates.

We are going to assume a number of things about the form of these equations, all of which can be fully justified, but which we will introduce more or less on the basis that they seem intuitively reasonable.

First, because the relative motion of the two reference frames is in the X direction, it is reasonable to expect that all distances measured at right angles to the X direction will be the same in both S and S', i.e.[5]

$$y = y' \text{ and } z = z'. \tag{21}$$

We now assume that (x, t) and (x', t') are related by the linear transformations

$$x' = Ax + Bt \tag{22}$$

$$t' = Cx + Dt. \tag{23}$$

Why linear? Assuming that space and time is homogeneous tells us that a linear relation is the only possibility[6]. What it amounts to saying is that it should not matter where in space we choose our origin of the spatial coordinates to be, not should it matter when we choose the orgin of time, i.e. the time that we choose to set as $t = 0$.

Now consider the origin O' of S'. This point is at $x' = 0$ which, if substituted into Eq. (22) gives

$$Ax + Bt = 0 \tag{24}$$

where x and t are the coordinates of O' as measured in S, i.e. at time t the origin O' has the X coordinate x, where x and t are related by $Ax + Bt = 0$. This can be written

$$\frac{x}{t} = -\frac{B}{A} \tag{25}$$

but x/t is just the velocity of the origin O' as measured in S. This origin will be moving at the same speed as the whole reference frame, so then we have

$$-\frac{B}{A} = v_x \tag{26}$$

which gives $B = -v_x A$ which can be substituted into Eq. (22) to give

$$x' = A(x - v_x t). \tag{27}$$

[5] If we assumed, for instance, that $z = kz'$, then it would also have to be true that $z' = kz$ if we reverse the roles of S and S', which tells us that $k^2 = 1$ and hence that $k = \pm 1$. We cannot have $z = -z'$ as the coordinate axes are clearly not 'inverted', so we must have $z = z'$.

[6] In general, x' will be a function of x and t, i.e. $x' = f(x, t)$ so that we would have $dx' = f_x dx + f_t dt$ where f_x is the partial derivatve of f with respect to x, and similarly for f_t. Homogeniety then means that these partial derivatives are constants. In other words, a small change in x and t produces the *same* change in x' no matter where in space or time the change takes place.

If we now solve Eq. (22) and Eq. (23) for x and t we get

$$x = \frac{Dx' + v_x At'}{AD - BC} \tag{28}$$

$$t = \frac{At' - Cx'}{AD - BC}. \tag{29}$$

If we now consider the origin O of the reference frame S, that is, the point $x = 0$, and apply the same argument as just used above, and noting that O will be moving with a velocity $-v_x$ with respect to S', we get

$$-\frac{v_x A}{D} = -v_x. \tag{30}$$

Comparing this with Eq. (26) we see that

$$A = D \tag{31}$$

and hence the transformations Eq. (28) and Eq. (29) from S' to S will be, after substituting for D and B:

$$\left.\begin{aligned} x &= \frac{A(x' + v_x t')}{A^2 + v_x AC} \\ t &= \frac{A(t' - (C/A)x')}{A^2 + v_x AC} \end{aligned}\right\} \tag{32}$$

which we can compare with the original transformation from S to S'

$$\left.\begin{aligned} x' &= A(x - v_x t) \\ t' &= A(t + (C/A)x). \end{aligned}\right\} \tag{33}$$

Any difference between the two transformation laws can only be due to the fact that the velocity of S' with respect to S is v_x and the velocity of S with respect to S' is $-v_x$[7]. So, given the transformation laws that give the S coordinates in terms of the S' coordinates, Eq. (32), the corresponding equations going the other way, Eq. (33), can be obtained simply swapping the primed and unprimed variables, and change the sign of v_x. If that is to be the case, then the factor $A^2 + v_x AC$ must be unity i.e.

$$A^2 + v_x AC = 1 \tag{34}$$

and it also suggests that C/A is proportional to v_x to guarantee the change in sign that occurs in passing from the expression for t to the one for t'. Thus,we have

$$A^2(1 + v_x(C/A)) = 1 \tag{35}$$

from which we get

$$A = \frac{1}{\sqrt{1 + v_x C/A}}. \tag{36}$$

[7]It is the assumed isotropy of space that comes into play here: there is no difference in the transformation laws relating the coordinates of an event in a reference frame to those of the event in a frame moving to the left or to the right, apart from a change in the sign of v_x.

If we now use the clue that C/A is proportional to v_x to try a substitution $C/A = -v_x/V^2$ where V is a quantity with the units of velocity yet to be determined, we have

$$A = \frac{1}{\sqrt{1-(v_x/V)^2}}. \tag{37}$$

so that finally the transformation laws become

$$\left.\begin{aligned} x' &= \frac{x - v_x t}{\sqrt{1-(v_x/V)^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (v_x/V^2)x}{\sqrt{1-(v_x/V)^2}}. \end{aligned}\right\} \tag{38}$$

This is a remarkable and very general result that depends purely on the assumed homogeneity and isotropy of space. At no stage have we mentioned light, or any other physical quantity for that matter, and yet we have been able to pin down the transformation laws relating coordinate systems for two different inertial frames of reference at least as far as there being only one undetermined quantity left, namely V. More information is needed to determine its value, but if we were to choose $V = \infty$, then we find that these transformation equations reduce to the Galilean transformation Eq. (1)! However, we have yet to make use of Einstein's second proposal. In doing so we are able to determine V, and find that V has an experimentally determinable, finite value.

To this end, let us suppose that when the two origins coincide, the clocks at O and O' both read zero, and also suppose that at that instant, a flash of light is emitted from the coincident points O and O'. In the frame of reference S this flash of light will be measured as lying on a spherical shell centred on O whose radius is growing at the speed c. However, by the second postulate, in the frame of reference S', the flash of light will also be measured as lying on a spherical shell centred on O' whose radius is also growing at the speed c. Thus, in S, if the spherical shell passes a point P with spatial coordinates (x, y, z) at time t, then by our definition of synchronization we must have:

$$x^2 + y^2 + z^2 = c^2 t^2$$

i.e.

$$x^2 + y^2 + z^2 - c^2 t^2 = 0. \tag{39}$$

The flash of light passing the point P in space at time t then defines an event with space-time coordinates (x, y, z, t). This event will have a different set of coordinates (x', y', z', t') relative to the frame of reference S' but by our definition of synchronization these coordinates must also satisfy:

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0. \tag{40}$$

We want to find how the two sets of coordinates (x, y, z, t) and (x', y', z', t') are related in order for both Eq. (39) and Eq. (40) to hold true. But we know quite generally that these coordinates must be related by the transformation laws Eq. (38) obtained above. If we substitute these expressions into Eq. (40) we get

$$\left[1-(cv_x/V^2)^2\right]x^2 + \left[1-(v_x/V)^2\right]y^2 + \left[1-(v_x/V)^2\right]z^2 - \left[1-(v_x/c)^2\right](ct)^2 - 2v_x\left[1-(c/V)^2\right]xt = 0. \tag{41}$$

This equation must reduce to Eq. (39). Either by working through the algebra, or simply by trial and error, it is straightforward to confirm that this requires $V = c$, i.e. the general transformation Eq. (38) with $V = c$, guarantees that the two spheres of light are expanding at the same rate, that is at the speed c, in both inertial frames of reference. Introducing a quantity γ defined by

$$\gamma = \frac{1}{\sqrt{1 - (v_x/c)^2}} \tag{42}$$

we are left with the final form of the transformation law consistent with light always being observed to be travelling at the speed c in all reference frames:

$$\left.\begin{aligned} x' &= \gamma(x - v_x t) \\ y' &= y \\ z' &= z \\ t' &= \gamma\big(t - (v_x/c^2)x\big). \end{aligned}\right\} \tag{43}$$

These are the equations of the Lorentz transformation. We can find the inverse transformation either by solving Eq. (43) for x, y, z, and t in terms of x', y', z', and t', or else by simply recognizing, as was mentioned above in the derivation of this transformation, that if S' is moving with velocity v_x relative to S, then S is moving with velocity $-v_x$ relative to S'. Consequently, all that is required is to exchange the primed and unprimed variables and change the sign of v_x in Eq. (43). The result by either method is

$$\left.\begin{aligned} x &= \gamma(x' + v_x t') \\ y &= y' \\ z &= z' \\ t &= \gamma\big(t' + (v_x/c^2)x'\big). \end{aligned}\right\} \tag{44}$$

These equations were first obtained by Lorentz who was looking for a mathematical transformation that left Maxwell's equations unchanged in form. However he did not assign any physical significance to his results. It was Einstein who first realized the true meaning of these equations, and consequently, with this greater insight, was able to derive them without reference at all to Maxwell's equations. The importance of his insight goes to the heart of relativity. Although the use of a flash of light played a crucial role in deriving the transformation equations, the final result simply establishes a connection between the two sets of space-time coordinates associated with a given event, this event being the passage of a flash of light past the point (x, y, z) at time t, as measured in S, or (x', y', z') at time t', as measured in S'. The transformation equations therefore represent a property that space and time must have in order to guarantee that light will always be observed to have the same speed c in all inertial frames of reference. But given that these transformation equations represent an intrinsic property of space and time, it can only be expected that the behaviour of other material objects, which may have nothing whatsoever to do with light, will also be influenced by this fundamental property of space and time. This is the insight that Einstein had, that the Lorentz transformation was saying something about the properties of space and time, and the consequent behaviour that matter and forces must have in order to be consistent with these properties.

Later we will see that the speed of light acts as an upper limit to how fast any material object can travel, be it light or electrons or rocket ships. In addition, we shall see that anything that travels at this speed c will always be observed to do so from all frames of reference. Light just happens to be one of the things in the universe that travels at this particular speed. Subatomic particles called neutrinos also apparently travel at the speed of light, so we could have formulated our arguments above on the basis of an expanding sphere of neutrinos! The constant c therefore represents a characteristic property of space and time, and only less significantly is it the speed at which light travels.

Two immediate conclusions can be drawn from the Lorentz Transformation. Firstly, suppose that , $v_x > c$ i.e. that S' is moving relative to S at a speed greater than the speed of light. In that case we find that $\gamma^2 < 0$ i.e. γ is imaginary so that both position and time in Eq. (43) become imaginary. However position and time are both physical quantities which must be measured as real numbers. In other words, the Lorentz transformation becomes physically meaningless if $v_x > c$. This immediately suggests that it is a physical impossibility for a material object to attain a speed greater than c relative to any reference frame S. The frame of reference in which such an object would be stationary will then also be moving at the speed v_x, but as we have just seen, in this situation the transformation law breaks down. We shall see later how the laws of dynamics are modified in special relativity, one of the consequences of this modification being that no material object can be accelerated to a speed greater than c[8].

Secondly, we can consider the form of the Lorentz Transformation in the mathematical limit $v_x << c$. We find that $\gamma \approx 1$ so that Eq. (43) becomes the equations of the Galilean Transformation, Eq. (1). (Though this also requires that the x dependent term in the time transformation equation to be negligible, which it will be over small enough distances). Thus, at low enough speeds, any unusual results due to the Lorentz transformation would be unobservable.

Perhaps the most startling aspect of the Lorentz Transformation is the appearance of a transformation for time. The result obtained earlier for the Galilean Transformation agrees with, indeed it was based on, our 'common sense' notion that time is absolute i.e. that time passes in a manner completely independent of the state of motion of any observer. This is certainly not the case with the Lorentz Transformation which leads, as we shall see, to the conclusion that moving clocks run slow. This effect, called time dilation, and its companion effect, length contraction will now be discussed.

9.1 Length Contraction

The first of the interesting consequences of the Lorentz Transformation is that length no longer has an absolute meaning: the length of an object depends on its motion relative to the frame of reference in which its length is being measured. Let us consider a rod moving with a velocity v_x relative to a frame of reference S, and lying along the X axis. This rod is then *stationary* relative to a frame of reference S' which is also moving with a velocity v_x relative to S.

[8] In principle there is nothing wrong with having an object that is initially travelling with a speed greater than c. In this case, c acts as a lower speed limit. Particles with this property, called tachyons, have be postulated to exist, but they give rise to problems involving causality (i.e. cause and effect) which make their existence doubtful.

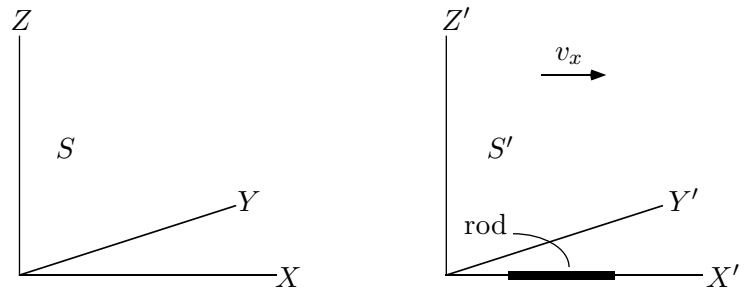

Figure 6: A rod of length at rest in reference frame S' which is moving with a velocity v_x with respect to another frame S.

As the rod is stationary in S', the ends of the rod will have coordinates x'_1 and x'_2 which remain fixed as functions of the time in S'. The length of the rod, as measured in S' is then

$$l_0 = x'_2 - x'_1 \tag{45}$$

where l_0 is known as the proper length of the rod i.e. l_0 is its length as measured in a frame of reference in which the rod is stationary. Now suppose that we want to measure the length of the rod as measured with respect to S. In order to do this, we measure the X coordinates of the two ends of the rod at the same time t, as measured by the clocks in S. Let x_2 and x_1 be the X coordinates of the two ends of the rod as measured in S at this time t. It is probably useful to be aware that we could rephrase the preceding statement in terms of the imaginary synchronized clocks introduced in Section 2.1 and Section 8 by saying that 'the two clocks positioned at x_2 and x_1 both read t when the two ends of the rod coincided with the points x_2 and x_1.' Turning now to the Lorentz Transformation equations, we see that we must have

$$\left.\begin{aligned} x'_1 &= \gamma(x_1 - v_x t)) \\ x'_2 &= \gamma(x_2 - v_x t). \end{aligned}\right\} \tag{46}$$

We then define the length of the rod as measured in the frame of reference S to be

$$l = x_2 - x_1 \tag{47}$$

where the important point to be re-emphasized is that this length is defined in terms of the positions of the ends of the rods as measured at the same time t in S. Using Eq. (46) and Eq. (47) we find

$$l_0 = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma l \tag{48}$$

which gives for l

$$l = \gamma^{-1} l_0 = \sqrt{1 - (v_x/c)^2}\, l_0. \tag{49}$$

But for $v_x < c$

$$\sqrt{1 - (v_x/c)^2} < 1 \tag{50}$$

so that

$$l < l_0. \tag{51}$$

Thus the length of the rod as measured in the frame of reference S with respect to which the rod is moving is shorter than the length as measured from a frame of reference S'

relative to which the rod is stationary. A rod will be observed to have its maximum length when it is stationary in a frame of reference. The length so-measured, l_0 is known as its *proper length*.

This phenomenon is known as the Lorentz-Fitzgerald contraction. It is not the consequence of some force 'squeezing' the rod, but it is a real physical phenomenon with observable physical effects. Note however that someone who actually looks at this rod as it passes by will not see a shorter rod. If the time that is required for the light from each point on the rod to reach the observer's eye is taken into account, the overall effect is that of making the rod appear as if it is rotated in space.

9.2 Time Dilation

Perhaps the most unexpected consequence of the Lorentz transformation is the way in which our 'commonsense' concept of time has to be drastically modified. Consider a clock C' placed at rest in a frame of reference S' at some point x' on the X axis. Suppose once again that this frame is moving with a velocity v_x relative to some other frame of reference S. At a time t'_1 registered by clock C' there will be a clock C_1 in the S frame of reference passing the position of C':

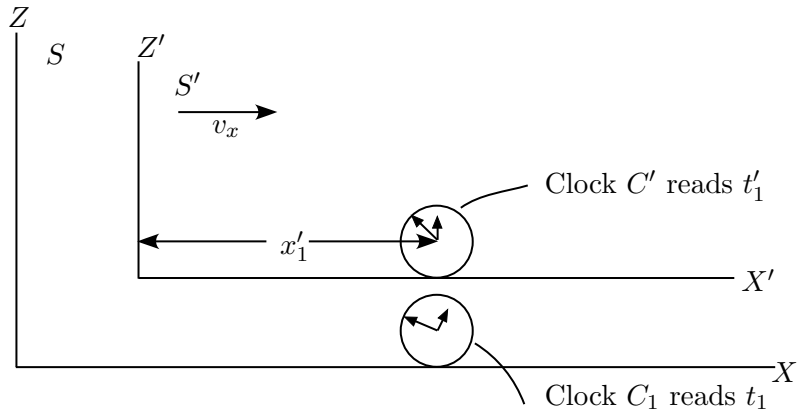

Figure 7: Clock C' stationary in S' reads t'_1 when it passes clock C_1 stationary in S, at which instant it reads t_1.

The time registered by C_1 will then be given by the Lorentz Transformation as

$$t_1 = \gamma(t'_1 + v_x x'/c^2). \tag{52}$$

Some time later, clock C' will read the time t'_2 at which instant a *different* clock C_2 in S will pass the position x'_1 in S'.

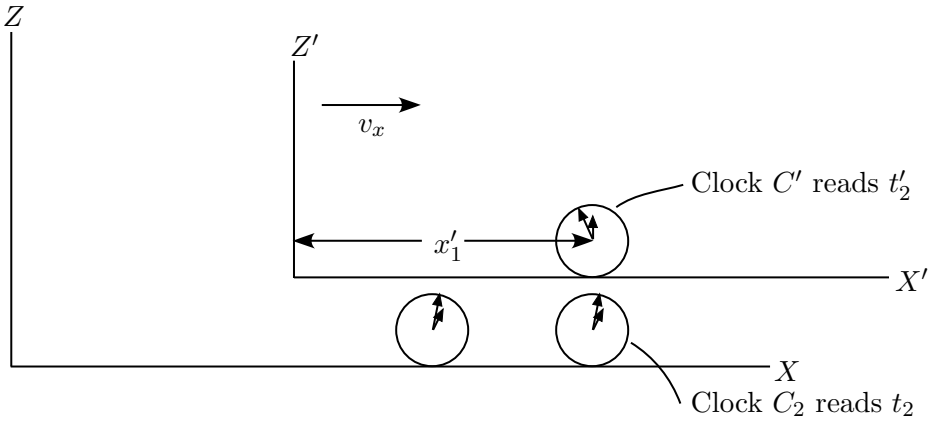

Figure 8: Clock C' stationary in S' reads t'_2 when it passes clock C_2 stationary in S, at which instant C_2 reads t_2.

This clock C_2 will read

$$t_2 = \gamma(t'_2 + v_x x'/c^2). \tag{53}$$

Thus, from Eq. (52) and Eq. (53) we have

$$\Delta t = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma \Delta t'. \tag{54}$$

Once again, since

$$\gamma = \frac{1}{\sqrt{1 - (v_x/c)^2}} > 1 \text{ if } v_x < c \tag{55}$$

we have

$$\Delta t > \Delta t'. \tag{56}$$

In order to interpret this result, suppose that $\Delta t'$ is the time interval between two 'ticks' of the clock C'. Then according to the clocks in S, these two 'ticks' are separated by a time interval Δt which, by Eq. (56) is $> \Delta t'$. Thus the time interval between 'ticks' is longer, as measured by the clocks in S, than what it is measured to be in S'. In other words, from the point of view of the frame of reference S, the clock (and all the clocks in S') are running slow. It appears from S that time is passing more slowly in S' than it is in S. This is the phenomenon of *time dilation*. A clock will be observed to run at its fastest when it is stationary in a frame of reference. The clock is then said to be measuring *proper time*.

This phenomenon is just as real as length contraction. One of its best known consequences is that of the increase in the lifetime of a radioactive particle moving at a speed close to that of light. For example, it has been shown that if the lifetime of a species of radioactive particle is measured while stationary in a laboratory to be T', then the lifetime of an identical particle moving relative to the laboratory is found to be given by $T = \gamma T'$, in agreement with Eq. (54) above.

Another well known consequence of the time dilation effect is the so-called twin or clock paradox. The essence of the paradox can be seen if we first of all imagine two clocks moving relative to each other which are synchronized when they pass each other. Then, in the frame of reference of one of the clocks, C say, the other clock will be measured as running slow, while in the frame of reference of clock C', the clock C will also be measured

to be running slow[9]. This is not a problem until one of the clocks does a U-turn in space (with the help of rocket propulsion, say) and returns to the position of the other clock. What will be found is that the clock that 'came back' will have lost time compared to the other. Why should this be so, as each clock could argue (if clocks could argue) that from its point of view it was the other clock that did the U-turn? The paradox can be resolved in many ways. The essence of the resolution, at least for the version of the clock paradox being considered here, is that there is not complete symmetry between the two clocks. The clock that turns back must have undergone acceleration in order to turn around. The forces associated with this acceleration will only be experienced by this one clock so that even though each clock could argue that it was the other that turned around and came back, it was only one clock that experienced an acceleration. Thus the two clocks have different histories between meetings and it is this asymmetry that leads to the result that the accelerated clock has lost time compared to the other. Of course, we have not shown how the turning around process results in this asymmetry: a detailed analysis is required and will not be considered here.

9.3 Simultaneity

Another consequence of the transformation law for time is that events which occur simultaneously in one frame of reference will not in general occur simultaneously in any other frame of reference. Thus, consider two events 1 and 2 which are simultaneous in S i.e. $t_1 = t_2$, but which occur at two different places x_1 and x_2. Then, in S', the time interval between these two events is

$$\begin{aligned} t_2' - t_1' &= \gamma(t_2 - v_x x_2/c^2) - \gamma(t_1 - v_x x_1/c^2) \\ &= \gamma(x_1 - x_2)v_x/c^2 \\ &\neq 0 \text{ as } x_1 \neq x_2. \end{aligned} \tag{57}$$

Here t_1' is the time registered on the clock in S' which coincides with the position x_1 in S at the instant t_1 that the event 1 occurs and similarly for t_2'. Thus events which appear simultaneous in S are not simultaneous in S'. In fact the order in which the two events 1 and 2 are found to occur in will depend on the sign of $x_1 - x_2$ or v_x. It is only when the two events occur at the same point (i.e. $x_1 = x_2$) that the events will occur simultaneously in all frames of reference.

9.4 Transformation of Velocities (Addition of Velocities)

Suppose, relative to a frame S, a particle has a velocity

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \tag{58}$$

where $u_x = dx/dt$ etc. What we require is the velocity of this particle as measured in the frame of reference S' moving with a velocity v_x relative to S. If the particle has coordinate x at time t in S, then the particle will have coordinate x' at time t' in S' where

$$x = \gamma()x' + v_x t') \text{ and } t = \gamma(t' + v_x x'/c^2). \tag{59}$$

[9]This appears to be paradoxical – how can *both* clocks consider the other as going slow? It should be borne in mind that the clocks C and C' are not being compared directly against one another, rather the time on each clock is being compared against the time registered on the collection of clocks that it passes in the other reference frame

If the particle is displaced to a new position $x + dx$ at time $t + dt$ in S, then in S' it will be at the position $x' + dx'$ at time $t' + dt'$ where

$$x + dx = \gamma \left(x' + dx' + v_x(t' + dt') \right)$$

$$t + dt = \gamma \left(t' + dt' + v_x(x' + dx')/c^2 \right)$$

and hence

$$dx = \gamma(dx' + v_x dt')$$

$$dt = \gamma(dt' + v_x dx'/c^2)$$

so that

$$u_x = \frac{dx}{dt} = \frac{dx' + v_x dt'}{dt' + v_x dx'/c^2}$$

$$= \frac{\frac{dx'}{dt'} + v_x}{1 + \frac{v_x}{c^2}\frac{dx'}{dt'}}$$

$$= \frac{u'_x + v_x}{1 + v_x u'_x/c^2} \tag{60}$$

where $u'_x = dx'/dt'$ is the X velocity of the particle in the S' frame of reference. Similarly, using $y = y'$ and $z = z'$ we find that

$$u_y = \frac{u'_y}{\gamma(1 + v_x u'_x/c^2)} \tag{61}$$

$$u_z = \frac{u'_z}{\gamma(1 + v_x u'_x/c^2)}. \tag{62}$$

The inverse transformation follows by replacing $v_x \to -v_x$ interchanging the primed and unprimed variables. The result is

$$\left.\begin{aligned} u'_x &= \frac{u_x - v_x}{1 - v_x u_x/c^2} \\ u'_y &= \frac{u_y}{\gamma(1 - v_x u'_x/c^2)} \\ u'_z &= \frac{u_z}{\gamma(1 - v_x u'_x/c^2)}. \end{aligned}\right\} \tag{63}$$

In particular, if $u_x = c$ and $u_y = u_z = 0$, we find that

$$u'_x = \frac{c - v_x}{1 - v_x/c} = c \tag{64}$$

i.e., if the particle has the speed c in S, it has the same speed c in S'. This is just a restatement of the fact that if a particle (or light) has a speed c in one frame of reference, then it has the same speed c in all frames of reference.

Now consider the case in which the particle is moving with a speed that is less that c, i.e. suppose $u_y = u_z = 0$ and $|u_x| < c$. We can rewrite Eq. (63) in the form

$$u'_x - c = \frac{u_x - c}{1 - u_x v_x/c^2} - c$$

$$= \frac{(c + v_x)(c - v_x)}{c(1 - v_x u_x/c^2)}. \tag{65}$$

Now, if S' is moving relative to S with a speed less than c, i.e. $|v_x| < c$, then along with $|u_x| < c$ it is not difficult to show that the right hand side of Eq. (65) is always negative i.e.

$$u'_x - c < 0 \text{ if } |u_x| < c,\ |v_x| < c \tag{66}$$

from which follows $u'_x < c$.

Similarly, by writing

$$u'_x + c = \frac{u_x - v_x}{1 - u_x v_x/c^2} + c$$

$$= \frac{(c + u_x)(c - v_x)}{c(1 - v_x u_x/c^2)} \tag{67}$$

we find that the right hand side of Eq. (67) is always positive provided $|u_x| < c$ and $|v_x| < c$ i.e.

$$u'_x + c > 0 \text{ if } |u_x| < c,\ |v_x| < c \tag{68}$$

from which follows $u'_x > -c$. Putting together Eq. (66) and Eq. (68) we find that

$$|u'_x| < c \text{ if } |u_x| < c \text{ and } |v_x| < c. \tag{69}$$

What this result is telling us is that if a particle has a speed less than c in one frame of reference, then its speed is always less than c in any other frame of reference, provided this other frame of reference is moving at a speed less than c. As an example, consider two objects A and B approaching each other, A at a velocity $u_x = 0.99c$ relative to a frame of reference S, and B stationary in a frame of reference S' which is moving with a velocity $v_x = -0.99c$ relative to S.

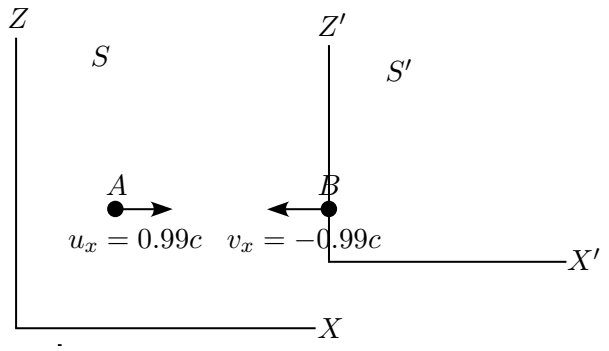

Figure 9: Object B stationary in reference frame S' which is moving with a velocity $v_x = -0.99c$ relative to reference frame S. Object A is moving with velocity $u_x = 0.99c$ with respect to reference frame S.

According to classical Newtonian kinematics, B will measure A as approaching at a speed of $1.98c$. However, according to the Einsteinian law of velocity addition, the velocity of A relative to B, i.e. the velocity of A as measured in frame S' is, from Eq. (63)

$$u'_x = \frac{0.99c - (-0.99c)}{1 + (0.99)^2} = 0.99995c$$

which is, of course, less than c, in agreement with Eq. (69).

In the above, we have made use of the requirement that all speeds be less than or equal to c. To understand physically why this is the case, it is necessary to turn to consideration of relativistic dynamics.

10 Relativistic Dynamics

Till now we have only been concerned with kinematics i.e. what we can say about the motion of the particle without consideration of its cause. Now we need to look at the laws that determine the motion i.e. the relativistic form of Newton's Laws of Motion. Firstly, Newton's First Law is accepted in the same form as presented in Section 2.2. However two arguments can be presented which indicate that Newton's Second Law may need revision. One argument only suggests that something may be wrong, while the second is of a much more fundamental nature. Firstly, according to Newton's Second Law if we apply a constant force to an object, it will accelerate without bound i.e. up to and then beyond the speed of light. Unfortunately, if we are going to accept the validity of the Lorentz Transformation, then we find that the factor γ becomes imaginary i.e. the factor γ becomes imaginary. Thus real position and time transform into imaginary quantities in the frame of reference of an object moving faster than the speed of light. This suggests that a problem exists, though it does turn out to be possible to build up a mathematical theory of particles moving at speeds greater than c (tachyons).

The second difficulty with Newton's Laws arise from the result, derived from the Second and Third laws, that in an isolated system, the total momentum of all the particles involved is constant, where momentum is defined, for a particle moving with velocity $\mathbf{u}$ and having mass m, by

$$\mathbf{p} = m\mathbf{u} \tag{70}$$

The question then is whether or not this law of conservation of momentum satisfies Einstein's first postulate, i.e. with momentum defined in this way, is momentum conserved in all inertial frames of reference? To answer this, we could study the collision of two bodies

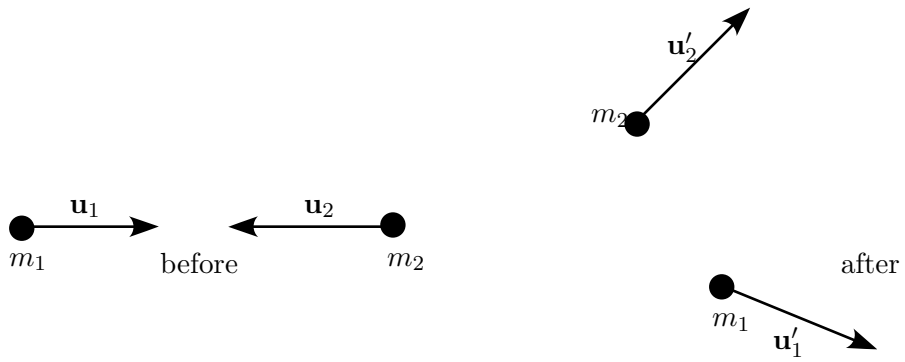

Figure 10: Collision between two particles used in discussing the conservation of momentum in different reference frames.

and investigate whether or not we always find that

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{u}_1' + m_2\mathbf{u}_2' \tag{71}$$

in *every inertial frame of reference*. Recall that the velocities must be transformed according to the relativistic laws given by Eq. (63). If, however, we retain the Newtonian principle that the mass of a particle is independent of the frame of reference in which it is measured (see Section 5) we find that Eq. (71) does *not* hold true in all frames of reference. Thus the Newtonian definition of momentum and the Newtonian law of conservation of momentum are inconsistent with the Lorentz transformation, even though at very low speeds (i.e. very much less than the speed of light) these Newtonian principles are known to yield results in agreement with observation to an exceedingly high degree of accuracy. So, instead of abandoning the momentum concept entirely in the relativistic theory, a more reasonable approach is to search for a generalization of the Newtonian concept of momentum in which the law of conservation of momentum is obeyed in all frames of reference. We do not know beforehand whether such a generalization even exists, and any proposals that we make can only be justified in the long run by the success or otherwise of the generalization in describing what is observed experimentally.

10.1 Relativistic Momentum

Any relativistic generalization of Newtonian momentum must satisfy two criteria:

1. Relativistic momentum must be conserved in all frames of reference.
2. Relativistic momentum must reduce to Newtonian momentum at low speeds.

The first criterion must be satisfied in order to satisfy Einstein's first postulate, while the second criterion must be satisfied as it is known that Newton's Laws are correct at sufficiently low speeds. By a number of arguments, the strongest of which being based on arguments concerning the symmetry properties of space and time, a definition for the relativistic momentum of a particle moving with a velocity $\mathbf{u}$ as measured with respect to a frame of reference S, that satisfies these criteria can be shown to take the form

$$\mathbf{p} = \frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{72}$$

where m_0 is the rest mass of the particle, i.e. the mass of the particle when at rest, and which can be identified with the Newtonian mass of the particle. With this form for the relativistic momentum, Einstein then postulated that, for a system of particles:

> *The total momentum of a system of particles is always conserved in all frames of reference, whether or not the total number of particles involved is constant.*

The above statement of the law of conservation of relativistic momentum generalized to apply to situations in which particles can stick together or break up (that is, be created or annihilated) is only a postulate whose correctness must be tested by experiment. However, it turns out that the postulate above, with relativistic momentum defined as in Eq. (72) is amply confirmed experimentally.

We note immediately that, for $u << c$, Eq. (72) becomes

$$\mathbf{p} = m_0\mathbf{u} \tag{73}$$

which is just the Newtonian form for momentum, as it should be.

It was once the practice to write the relativistic momentum, Eq. (72), in the form

$$\mathbf{p} = m\mathbf{u} \tag{74}$$

where

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{75}$$

which leads us to the idea that the mass of a body (m) increases with its velocity. However, while a convenient interpretation in certain instances, it is not a recommended way of thinking in general since the (velocity dependent) mass defined in this way does not always behave as might be expected. It is better to consider m_0 as being an intrinsic property of the particle (in the same way as its charge would be), and that it is the momentum that is increased by virtue of the factor in the denominator in Eq. (72).

Having now defined the relativistic version of momentum, we can now proceed towards setting up the relativistic ideas of force, work, and energy.

10.2 Relativistic Force, Work, Kinetic Energy

All these concepts are defined by analogy with their corresponding Newtonian versions. Thus relativistic force is defined as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{76}$$

a definition which reduces to the usual Newtonian form at low velocities. This force will do work on a particle, and the *relativistic work* done by $\mathbf{F}$ during a small displacement $d\mathbf{r}$ is, once again defined by analogy as

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{77}$$

The rate at which $\mathbf{F}$ does work is then

$$P = \mathbf{F} \cdot \mathbf{u} \tag{78}$$

and we can introduce the notion of relativistic kinetic energy by viewing the work done by $\mathbf{F}$ as contributing towards the kinetic energy of the particle i.e.

$$P = \frac{dT}{dt} = \mathbf{F} \cdot \mathbf{u} \tag{79}$$

where T is the *relativistic kinetic energy* of the particle. We can write this last equation as

$$\begin{aligned}
\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{u} &= \mathbf{u} \cdot \frac{d\mathbf{p}}{dt} \\
&= \mathbf{u} \cdot \frac{d}{dt} \frac{m_0 \mathbf{u}}{\sqrt{1 - u^2/c^2}} \\
&= \frac{m_0 \mathbf{u} \cdot \dfrac{d\mathbf{u}}{dt}}{\sqrt{1 - u^2/c^2}} + \frac{m_0 \mathbf{u} \cdot \mathbf{u} u \dfrac{du}{dt}}{c^2 \sqrt{1 - u^2/c^2}}
\end{aligned}$$

But

$$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = u\frac{du}{dt} \tag{80}$$

and hence

$$\frac{dT}{dt} = \left[\frac{m_0}{\sqrt{1-u^2/c^2}} + \frac{m_0 u^2/c^2}{\sqrt{(1-u^2/c^2)^3}}\right] u\frac{du}{dt}$$

$$= \frac{m_0}{\sqrt{(1-u^2/c^2)^3}} u\frac{du}{dt}$$

so that we end up with

$$\frac{dT}{dt} = \frac{d}{dt}\left[\frac{m_0 c^2}{\sqrt{1-u^2/c^2}}\right]. \tag{81}$$

Integrating with respect to t gives

$$T = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} + \text{ constant.} \tag{82}$$

By requiring that $T = 0$ for $u = 0$, we find that

$$T = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} - m_0 c^2. \tag{83}$$

Interestingly enough, if we suppose that $u << c$, we find that, by the binomial approximation[10]

$$\frac{1}{\sqrt{1-u^2/c^2}} = (1-u^2/c^2)^{-\frac{1}{2}} \approx 1 + \frac{u^2}{2c^2} \tag{84}$$

so that

$$T \approx m_0 c^2(1+u^2/c^2) - m_0 c^2 \approx \tfrac{1}{2} m_0 c^2 \tag{85}$$

which, as should be the case, is the classical Newtonian expression for the kinetic energy of a particle of mass moving with a velocity $\mathbf{u}$.

10.3 Total Relativistic Energy

We can now define a quantity E by

$$E = T + m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}. \tag{86}$$

This quantity E is known as the *total relativistic energy* of the particle of rest mass m_0. It is all well and good to define such a thing, but, apart from the neatness of the expression, is there any real need to introduce such a quantity? In order to see the value of defining the total relativistic energy, we need to consider the transformation of momentum between different inertial frames S and S'. To this end consider

$$p_x = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} \tag{87}$$

where

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{88}$$

[10]The binomial approximation is $(1+x)^n \approx 1 + nx$ if $x << 1$.

and where $\mathbf{u}$ is the velocity of the particle relative to the frame of reference S. In terms of the velocity $\mathbf{u}'$ of this particle relative to the frame of reference S' we can write

$$u_x = \frac{u'_x + v_x}{1 + u'_x v_x/c^2} \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v_x/c^2)} \quad u_z = \frac{u'_z}{\gamma(1 + u'_x v_x/c^2)} \tag{89}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v_x^2/c^2}} \tag{90}$$

as before. After a lot of exceedingly tedious algebra, it is possible to show that

$$\sqrt{1 - u^2/c^2} = \frac{\sqrt{1 - u'^2/c^2}\sqrt{1 - v_x^2/c^2}}{1 + u'_x v_x/c^2} \tag{91}$$

so that, using Eq. (89), Eq. (90) and Eq. (91) we find

$$\begin{aligned} p_x &= \frac{m_0(u'_x + v_x)}{\sqrt{(1 - u'^2/c^2)(1 - v_x^2/c^2)}} \\ &= \gamma\left[\frac{m_0 u'_x}{\sqrt{1 - u'^2/c^2}} + v_x\Big(\frac{m_0}{\sqrt{1 - u'^2/c^2}}\Big)\right] \end{aligned}$$

which we can readily write as

$$p_x = \gamma\big[p'_x + v_x(E'/c^2)\big] \tag{92}$$

i.e. we see appearing the total energy E' of the particle as measured in S'.

A similar calculation for p_y and p_z yields

$$p_y = p'_y \text{ and } p_z = p'_z \tag{93}$$

while for the energy E we find

$$\begin{aligned} E &= \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} \\ &= \frac{m_0 c^2}{\sqrt{1 - u'^2/c^2}} \cdot \frac{1 + u'_x v_x/c^2}{\sqrt{1 - v_x^2/c^2}} \\ &= \gamma\Big[\frac{m_0 c^2}{\sqrt{1 - u'^2/c^2}} + \frac{m_0 u'_x v_x}{\sqrt{1 - u'^2/c^2}}\Big] \end{aligned}$$

which we can write as

$$E = \gamma\left[E' + p'_x v_x\right]. \tag{94}$$

Now consider the collision between two particles 1 and 2. Let the X components of momentum of the two particles be p_{1x} and p_{2x} relative to S. Then the total momentum in S is

$$P_x = p_{1x} + p_{2x} \tag{95}$$

where P_x is, by conservation of relativistic momentum, a constant, i.e. P_x stays the same before and after any collision between the particles. However

$$p_{1x} + p_{2x} = \gamma\left(p'_{1x} + p'_{2x}\right) + \gamma\left(E'_1 + E'_2\right) v_x/c^2 \tag{96}$$

where p'_{1x} and p'_{2x} are the X component of momentum of particles 1 and 2 respectively, while E'_1 and E'_2 are the energies of particles 1 and 2 respectively, all relative to frame of reference S'. Thus we can write

$$P_x = \gamma P'_x + \gamma \left(E'_1 + E'_2\right) v_x/c^2. \tag{97}$$

Once again, as momentum is conserved in all inertial frames of reference, we know that P'_x is also a constant i.e. the same before and after any collision. Thus we can conclude from Eq. (97) that

$$E'_1 + E'_2 = \text{ constant} \tag{98}$$

i.e. the total relativistic energy in S' is conserved. But since S' is an arbitrary frame of reference, we conclude that the total relativistic energy is conserved in all frames of reference (though of course the conserved value would in general be different in different frames of reference). Since, as we shall see later, matter can be created or destroyed, we generalize this to read:

> *The total relativistic energy of a system of particles is always conserved in all frames of reference, whether or not the total number of particles remains a constant.*

Thus we see that conservation of relativistic momentum implies conservation of total relativistic energy in special relativity whereas in Newtonian dynamics, they are independent conditions. Nevertheless, both conditions have to be met in when determining the outcome of any collision between particles, i.e. just as in Newtonian dynamics, the equations representing the conservation of energy and momentum have to be employed.

A useful relationship between energy and momentum can also be established. Its value lies both in treating collision problems and in suggesting the existence of particles with zero rest mass. The starting point is the expression for energy

$$E = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} \tag{99}$$

from which we find

$$E^2 = \frac{m_0^2 c^4}{1 - u^2/c^2}$$

$$= \frac{m_0 c^4 \left[1 - u^2/c^2 + u^2/c^2\right]}{1 - u^2/c^2}$$

so that

$$E^2 = m_0^2 c^4 + \frac{m_0 u^2}{1 - u^2/c^2} \cdot c^2. \tag{100}$$

But

$$\mathbf{p} = \frac{m_0 \mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

and hence

$$p^2 = \mathbf{p} \cdot \mathbf{p} = \frac{m_0 u^2}{1 - u^2/c^2}$$

which can be combined with Eq. (100) to give

$$E^2 = p^2 c^2 + m_0^2 c^4. \tag{101}$$

We now will use the above concept of relativistic energy to establish the most famous result of special relativity, the equivalence of mass and energy.

10.4 Equivalence of Mass and Energy

This represents probably the most important result of special relativity, and gives a deep physical meaning to the concept of the total relativistic energy E. To see the significance of E in this regard, consider the breakup of a body of rest mass m_0 into two pieces of rest masses m_{01} and m_{02}:

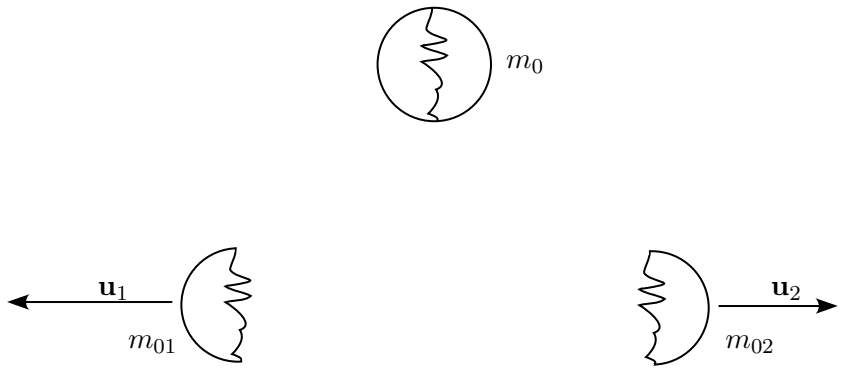

Figure 11: Break up of a body of rest mass m_0 into two parts of rest masses m_{01} and m_{02}, moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ relative to the rest frame of the original object.

We could imagine that the original body is a radioactive nucleus, or even simply two masses connected by a coiled spring. If we suppose that the initial body is stationary in some frame S, and the debris flies apart with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ relative to S then, by the conservation of energy in S:

$$\begin{aligned} E = m_0c^2 &= E_1 + E_2 \\ &= \frac{m_{01}c^2}{\sqrt{1-u_1^2/c^2}} \frac{m_{02}c^2}{\sqrt{1-u_2^2/c^2}} \end{aligned}$$

so that

$$\begin{aligned} (m_0 - m_{01} - m_{02})c^2 =& m_{01}c^2\Big[\frac{1}{\sqrt{1-u_1^2/c^2}} - 1\Big] \\ &+ m_{02}c^2\Big[\frac{1}{\sqrt{1-u_2^2/c^2}} - 1\Big] \\ =& T_1 + T_2 \end{aligned} \tag{102}$$

where T_1 and T_2 are the relativistic kinetic energies of the two masses produced. Quite obviously, T_1 and $T_2 > 0$ since

$$\frac{1}{\sqrt{1-u_1^2/c^2}} - 1 > 0 \tag{103}$$

and similarly for the other term and hence

$$m_0 - m_{01} - m_{02} > 0 \tag{104}$$

or

$$m_0 < m_{01} + m_{02}. \tag{105}$$

What this result means is that the total rest mass of the two separate masses is less than that of the original mass. The difference, Δm say, is given by

$$\Delta m = \frac{T_1 + T_2}{c^2}. \tag{106}$$

We see therefore that part of the rest mass of the original body has disappeared, and an amount of kinetic energy given by Δmc^2 has appeared. The inescapable conclusion is that some of the rest mass of the original body has been converted into the kinetic energy of the two masses produced.

The interesting result is that none of the masses involved need to be travelling at speeds close to the speed of light. In fact, Eq. (106) can be written, for $u_1, u_2 << c$ as

$$\Delta m = \frac{\frac{1}{2}m_{01}u_1^2 + \frac{1}{2}m_{02}u_2^2}{c^2} \tag{107}$$

so that only classical Newtonian kinetic energy appears. Indeed, in order to measure the mass loss Δm, it would be not out of the question to bring the masses to rest in order to determine their rest masses. Nevertheless, the truly remarkable aspect of the above conclusions is that it has its fundamental origin in the fact that there exists a universal maximum possible speed, the speed of light which is built into the structure of space and time, and this structure ultimately exerts an effect on the properties of matter occupying space and time, that is, its mass and energy.

The reverse can also take place i.e. matter can be created out of energy as in, for instance, a collision between particles having some of their energy converted into new particles as in the proton-proton collision

$$p + p \rightarrow p + p + p + \bar{p}$$

where a further proton and antiproton ($\bar{p}$) have been produced.

A more mundane outcome of the above connection between energy and mass is that rather than talking about the rest mass of a particle, it is often more convenient to talk about its rest energy. A particle of rest mass m_0 will, of course, have a rest energy m_0c^2. Typically the rest energy (or indeed any energy) arising in atomic, nuclear, or elementary physics is given in units of electron volts. One electron volt (eV) is the energy gained by an electron accelerated through a potential difference of 1 volt i.e.

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ Joules}.$$

An example of the typical magnitudes of the rest energies of elementary particles is that of the proton. With a rest mass of $m_p = 1.67 \times 10^{-33}$ kg, the proton has a rest energy of

$$m_pc^2 = 938.26 \text{ MeV}.$$

10.5 Zero Rest Mass Particles

For a single particle, rest mass m_0, its momentum p and energy E are related by the expression:

$$E^2 = p^2c^2 + M_0^2c^4.$$

This result allows us to formally take the limit of $m_0 \rightarrow 0$ while keeping E and p fixed. The result is a relationship between energy and momentum for a particle of zero rest mass. In this limit, with $E, p \neq 0$, we have

$$E = pc = |\mathbf{p}|c \tag{108}$$

i.e. p is the magnitude of the momentum vector $\mathbf{p}$. If we rearrange Eq. (86) to read

$$E\sqrt{1-u^2/c^2}$$

and if we then let $m_0 \to 0$ with $E \neq 0$, we must have

$$\sqrt{1-u^2/c^2} \to 0$$

so that, in the limit of $m_0 \to 0$, we find that

$$u = c. \tag{109}$$

Thus, if there exists particles of zero rest mass, we see that their energy and momentum are related by Eq. (108) and that they always travel at the speed of light. Particles with zero rest mass need not exist since all that we have presented above is a mathematical argument. However it turns out that they do indeed exist: the photon (a particle of light) and the neutrino, though recent research in solar physics seems to suggest that the neutrino may in fact have a non-zero, but almost immeasurably tiny mass. Quantum mechanics presents us with a relationship between frequency f of a beam of light and the energy of each photon making up the beam:

$$E = hf = \hbar\omega \tag{110}$$

11 Geometry of Spacetime

The theory of relativity is a theory of space and time and as such is a geometrical theory, though the geometry of space and time together is quite different from the Euclidean geometry of ordinary 3-dimensional space. Nevertheless it is found that if relativity is recast in the language of vectors and "distances" (or preferably "intervals") a much more coherent picture of the content of the theory emerges. Indeed, relativity is seen to be a theory of the geometry of the single entity, 'spacetime', rather than a theory of space and time. Furthermore, without the geometrical point-of-view it would be next to impossible to extend special relativity to include transformations between arbitrary (non-inertial) frames of reference, which ultimately leads to the general theory of relativity, the theory of gravitation. In order to set the stage for a discussion of the geometrical properties of space and time, a brief look at some of the more familiar ideas of geometry, vectors etc in ordinary three dimensional space is probably useful.

11.1 Geometrical Properties of 3 Dimensional Space

For the present we will not be addressing any specifically relativistic problem, but rather we will concern ourselves with the issue of fixing the position in space of some arbitrary point. To do this we could, if we wanted to, imagine a suitable set of rulers so that the position of a point P can be specified by the three coordinates (x, y, z) with respect to this coordinate system, which we will call R.

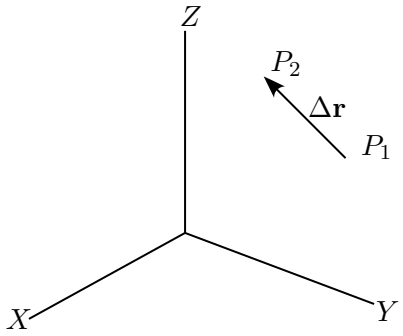

Figure 12: A displacement vector $\Delta\mathbf{r}$ in space with an arbitrary coordinate system R.

If we then consider two such points P_1 with coordinates (x_1, y_1, z_1) and P_2 with coordinates (x_2, y_2, z_2) then the line joining these two points defines a vector $\Delta\mathbf{r}$ which we can write in component form with respect to R as

$$\Delta\mathbf{r} \doteq \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}_R \tag{111}$$

where the subscript R is to remind us that the components are specified relative to the set of coordinates R. Why do we need to be so careful? Obviously, it is because we could have, for instance, used a different set of axes R' which have been translated and rotated relative to the first:

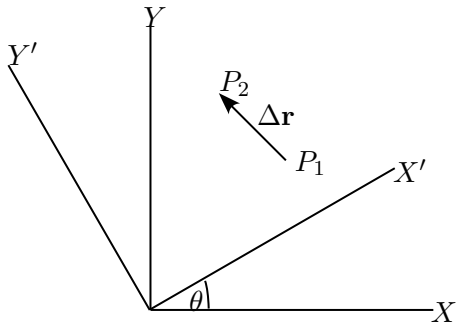

Figure 13: Displacement vector and two coordinate systems rotated with respect to each other about Z axis through angle θ. The vector has an existence independent of the choice of coordinate systems.

In this case the vector $\Delta\mathbf{r}$ will have new components, but the vector itself will *still be the same vector* i.e.

$$\Delta\mathbf{r} \doteq \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}_R \doteq \begin{pmatrix} x_2' - x_1' \\ y_2' - y_1' \\ z_2' - z_1' \end{pmatrix}_{R'} \tag{112}$$

or

$$\Delta\mathbf{r} \doteq \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_R \doteq \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}_{R'} \tag{113}$$

So the components themselves are meaningless unless we know with respect to what coordinate system they were determined. In fact, the lack of an absolute meaning of the components unless the set of axes used is specified means that the vector $\Delta\mathbf{r}$ is not so

much 'equal' to the column vector as 'represented by' the column vectors – hence the use of the dotted equal sign '$\doteq$' to indicate 'represented by'.

The description of the vector in terms of its components relative to some coordinate system is something done for the sake of convenience. Nevertheless, although the components may change as we change coordinate systems, what does not change is the vector itself, i.e. it has an existence independent of the choice of coordinate system. In particular, the length of $\Delta\mathbf{r}$ and the angles between any two vectors $\Delta\mathbf{r}_1$ and $\Delta\mathbf{r}_2$ will be the same in any coordinate system.

While these last two statements may be obvious, it is important for what comes later to see that they also follow by explicitly calculating the length and angle between two vectors using their components in two different coordinate systems. In order to do this we must determine how the coordinates of $\Delta\mathbf{r}$ are related in the two different coordinate systems. We can note that the displacement of the two coordinate systems with respect to each other is immaterial as we are considering differences between vectors thus we only need to worry about the rotation which we have, for simplicity, taken to be through an angle θ about the Z axis (see the above diagram). The transformation between the sets of coordinates can then be shown to be given, in matrix form, by

$$\begin{pmatrix}\Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix}_{R'} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\end{pmatrix}_{R} \tag{114}$$

Using this transformation rule, we can show that

$$(\Delta x)^2+(\Delta y)^2+(\Delta z)^2=(\Delta x')^2+(\Delta y')^2+(\Delta z')^2 \tag{115}$$

where each side of this equation is, obviously, the (distance)^2 between the points P_1 and P_2. Further, for any two vectors $\Delta\mathbf{r}_1$ and $\Delta\mathbf{r}_2$ we find that

$$\Delta x_1\Delta x_2+\Delta y_1\Delta y_2+\Delta z_1\Delta z_2=\Delta x_1'\Delta x_2'+\Delta y_1'\Delta y_2'+\Delta z_1'\Delta z_2' \tag{116}$$

where each side of the equation is the scalar product of the two vectors i.e. $\Delta\mathbf{r}_1\cdot\Delta\mathbf{r}_2$. This result tells us that the angle between is the same in both coordinate systems. Thus the transformation Eq. (114) is consistent with the fact that the length and relative orientation of these vectors is independent of the choice of coordinate systems, as it should be.

It is at this point that we turn things around and say that *any* quantity that has three components that transform in exactly the same way as $\Delta\mathbf{r}$ under a rotation of coordinate system constitutes a *three-vector*. An example is force, for which

$$\begin{pmatrix}F_x'\\ F_y'\\ F_z'\end{pmatrix}_{R'} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}F_x\\ F_y\\ F_z\end{pmatrix}_{R} \tag{117}$$

for two coordinate systems R and R' rotated relative to each other by an angle θ about the Z-axis. Other three-vectors are electric and magnetic fields, velocity, acceleration etc. Since the transformation matrix in Eq. (117) is identical to that appearing in Eq. (114), any three-vector is guaranteed to have the same length (i.e. magnitude) and orientation irrespective of the choice of coordinate system. In other words we can claim that such a three vector has an absolute meaning independent of the choice of coordinate system used to determine its components.

11.2 Space Time Four Vectors

What we do now is make use of the above considerations to introduce the idea of a vector to describe the separation of two events occurring in spacetime. The essential idea is to show that the coordinates of an event have transformation properties analogous to Eq. (114) for ordinary three-vectors, though with some surprising differences. To begin, we will consider two events E_1 and E_2 occurring in spacetime. For event E_1 with coordinates (x_1, y_1, z_1, t_1) in frame of reference S and (x'_1, y'_1, z'_1, t'_1) in S', these coordinates are related by the Lorentz transformation which we will write as

$$\left.\begin{aligned} ct'_1 &= \gamma ct_1 - \frac{\gamma v_x}{c} x_1 \\ x'_1 &= -\frac{\gamma v_x}{c} ct_1 + \gamma x_1 \\ y'_1 &= y_1 \\ z'_1 &= z_1 \end{aligned}\right\} \tag{118}$$

and similarly for event E_2. Then we can write

$$\left.\begin{aligned} c\Delta t' = c(t'_2 - t'_1) &= \gamma c\Delta t - \frac{\gamma v_x}{c}\Delta x \\ \Delta x' = x'_2 - x'_1 &= -\frac{\gamma v_x}{c} c\Delta t + \gamma \Delta x \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned}\right\} \tag{119}$$

which we can write as

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}_{S'} = \begin{pmatrix} \gamma & -\gamma v_x/c & 0 & 0 \\ -\gamma v_x/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{S} . \tag{120}$$

It is tempting to interpret this equation as relating the components with respect to a coordinate system S' of some sort of 'vector', to the components with respect to some other coordinate system S, of the same vector. We would be justified in doing this if this 'vector' has the properties, analogous to the length and angle between vectors for ordinary three-vectors, which are independent of the choice of reference frame. It turns out that it is 'length' defined as

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta y)^2\right] = (c\Delta t)^2 - (\Delta \mathbf{r})^2 \tag{121}$$

that is invariant for different reference frames i.e.

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] = (c\Delta t')^2 - \left[(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2\right] \tag{122}$$

This invariant quantity Δs is known as the *interval* between the two events E_1 and E_2. Obviously Δs is analogous to, but fundamentally different from, the length of a three-vector in that it can be positive, zero, or negative. We could also talk about the 'angle' between two such 'vectors' and show that

$$(c\Delta t_1)(c\Delta t_2) - [\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1)\Delta z_2] \tag{123}$$

has the same value in all reference frames. This is analogous to the scalar product for three-vectors. The quantity defined by

$$\Delta\vec{s} = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{124}$$

is then understood to correspond to a property of spacetime representing the separation between two events which has an absolute existence independent of the choice of reference frame, and is known as a *four-vector*. In order to distinguish a four-vector from an ordinary three-vector, a superscript arrow will be used.

As was the case with three-vectors, any quantity which transforms in the same way as $\Delta\vec{s}$ is also termed a four-vector. For instance, we have shown that

$$\left.\begin{aligned} E'/c &= \gamma(E/c) - \frac{\gamma v_x}{c} p_x \\ p'_x &= -\frac{\gamma v_x}{c}(E/c) + \gamma p_x \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned}\right\} \tag{125}$$

which we can write as

$$\begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v_x/c & 0 & 0 \\ -\gamma v_x/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \tag{126}$$

where we see that the same matrix appears on the right hand side as in the transformation law for $\Delta\vec{s}$. This expression relates the components, in two different frames of reference S and S', of the four-momentum of a particle. This four-momentum is, of course, by virtue of this transformation property, also a four-vector. We can note that the ('length')^2 of this four-vector is given by

$$(E/c)^2 - \left[p_x^2 + p_y^2 + p_z^2\right] = (E/c)^2 - \mathbf{p}^2 = (E^2 - \mathbf{p}^2c^2)/c^2 = m_0^2c^2 \tag{127}$$

where m_0 is the rest mass of the particle. This quantity is the same (i.e. invariant) in different frames of reference.

A further four-vector is the velocity four-vector

$$\vec{v} \doteq \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} \tag{128}$$

where

$$d\tau = ds/c \tag{129}$$

and is known as the *proper time interval*. This is the time interval measured by a clock in its own rest frame as it makes its way between the two events an interval ds apart. The invariant ('length')^2 of the velocity four-vector is just c^2.

To emphasize the vector nature of the four-vector quantities introduced above, it is usual to introduce a more uniform way of naming the components, as follows:

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z \tag{130}$$

where the superscript numbers are NOT powers of x. Similarly, for the components of the momentum four-vector we have

$$p^0 = E/c, \qquad p^1 = p_x, \qquad p^2 = p_y, \qquad p^3 = p_z. \tag{131}$$

In terms of these names for the components we can write the Lorentz transformation equations as

$$(\Delta x^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu_\nu \Delta x^\nu \tag{132}$$

where Λ^μ_ν are the components of the 4×4 matrix appearing in Eq. (120) and Eq. (126). This expression is more usually written in the form

$$(\Delta x^\mu)' = \Lambda^\mu_\nu \Delta x^\nu \tag{133}$$

where the summation over ν is understood. This is the so-called Einstein summation convention, and it is implied whenever two indices occur in a 'one up-one down' combination in a product. Thus here, as the index ν appears 'down' in Λ^μ_ν and 'up' in Δx^ν, a summation over this index is understood.

In terms of this convention, the interval Δs can be written

$$(\Delta s)^2 = g_{\mu\nu} \Delta x^\mu \Delta^\nu \tag{134}$$

where there appears a new term $g_{\nu\mu}$, the components of a matrix known as the metric tensor, with the values

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1 \tag{135}$$

and all other components zero. Usually the brackets around $(\Delta s)^2$ are dropped so that equation Eq. (134) becomes

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{136}$$

It is this metric tensor which plays a central role in general relativity. There the components of $g_{\mu\nu}$ are not simple constants but rather are functions of the spacetime coordinates x^μ. In this way the curved nature of spacetime is taken into account when determining the interval between two events.

11.3 Spacetime Diagrams

Till now we have represented a frame of reference S by a collection of clocks and rulers. An alternative way of doing the same thing is to add a fourth axis, the time axis, 'at right angles' to the X, Y, Z axes. On this time axis we can plot the time t that the clock reads at the location of an event. Obviously we cannot draw in such a fourth axis, but we can suppress the Y, Z coordinates for simplicity and draw as in Fig. (14):

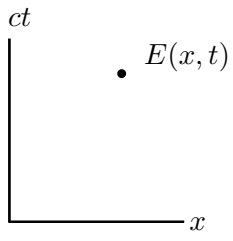

Figure 14: An event represented as a point in spacetime.

This representation is known as a spacetime or Minkowski diagram and on it we can plot the positions in space and time of the various events that occur in spacetime. In particular we can plot the motion of a particle through space and time. The curve traced out is known as the world line of the particle. We can note that the slope of such a world line must be greater than the slope of the world line of a photon since all material particles move with speeds less than the speed of light. Some typical world lines are illustrated in Fig. (15) below.

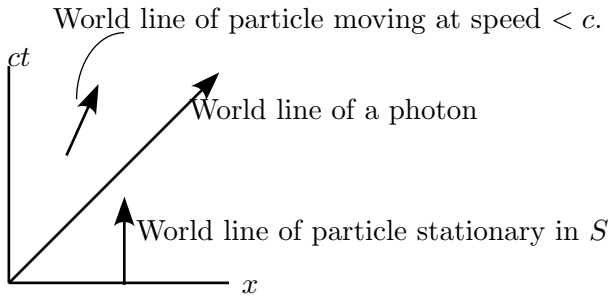

Figure 15: Diagram illustrating different kinds of world lines.

The above diagram gives the coordinates of events as measured in a frame of reference S say. We can also use these spacetime diagrams to illustrate Lorentz transformations from one frame of reference to another. Unfortunately, due to the peculiar nature of the interval between two events in spacetime, the new set of axes for some other frame of reference S' is not a simple rotation of the old axes. It turns out that these new axes are oblique, as illustrated below, and with increasing speeds of S' relative to S, these axes close in on the world line of the photon passing through the common origin.

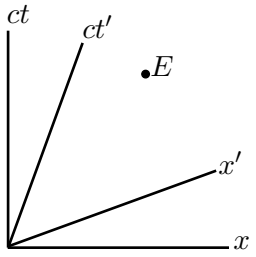

Figure 16: Space and time axes for two different reference frames. The rectilinear axes are for the reference frame S, the oblique axes those for a reference frame S' moving with respect to S.

We will not be considering this aspect of spacetime diagrams here. However, what we will briefly look at is some of the properties of the spacetime interval Δs that leads to this strange behaviour.

11.4 Properties of Spacetime Intervals

We saw in the preceding section that one of the invariant quantities is the interval Ds as it is just the "length" of the four-vector. As we saw earlier, it is the analogue in spacetime of the familiar distance between two points in ordinary 3-dimensional space. However, unlike the ordinary distance between two points, or more precisely (distance)^2, which is always positive (or zero), the interval between two events E_1 and E_2 i.e. Δs^2, can be positive, zero, or negative. The three different possibilities have their own names:

1. $\Delta s^2 < 0$: E_1 and E_2 are separated by a *space-like* interval.
2. $\Delta s^2 = 0$: E_1 and E_2 are separated by a *light-like* interval.
3. $\Delta s^2 > 0$: E_1 and E_2 are separated by a *time-like* interval.

What these different possibilities represent is best illustrated on a spacetime diagram. Suppose an event O occurs at the spacetime point $(0, 0)$ in some frame of reference S. We can divide the spacetime diagram into two regions as illustrated in the figure below: the shaded region lying between the world lines of photons passing through $(0, 0)$, and the unshaded region lying outside these world lines. Note that if we added a further space axis, in the Y direction say, the world lines of the photons passing through will lie on a cone with its vertex at O. This cone is known as the 'light cone'. Then events such as Q will lie 'inside the light cone', events such as P 'outside the light cone', and events such as R 'on the light cone'.

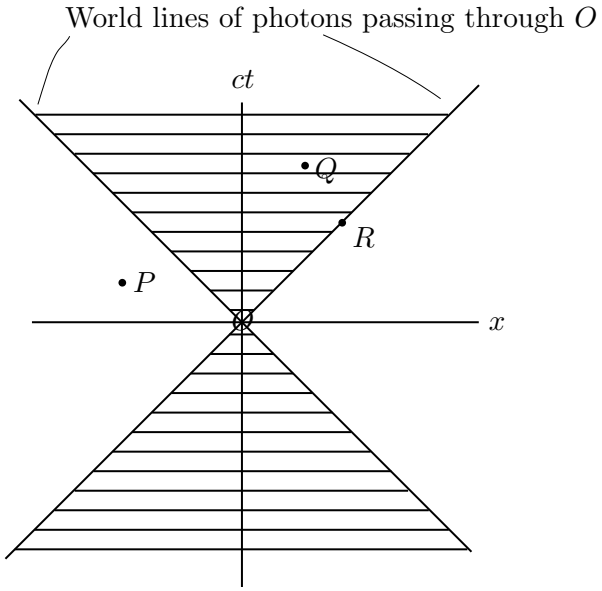

Figure 17: The point Q within the light cone (the shaded region) is separated from O by a time-like interval. A signal travelling at a speed less than c can reach Q from O. The point R on the edge of the light cone is separated from O by a light-like interval, and a signal moving at the speed c can reach R from O. The point P is outside the light cone. No signal can reach P from O.

Consider now the sign of Δs^2 between events O and P. Obviously

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 < 0 \tag{137}$$

i.e. all points outside the light cone through O are separated from O by a space-like interval. Meanwhile, for the event Q we have

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 > 0 \tag{138}$$

i.e. all points inside the light cone through O are separated from O by a time-like interval. Finally for R we have

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 = 0 \tag{139}$$

i.e. all points on the light cone through O are separated from O by a light-like interval.

The physical meaning of these three possibilities can be seen if we consider whether or not the event O can in some way affect the events P, Q, or R. In order for one event to physically affect another some sort of signal must make its way from one event to the other. This signal can be of any kind: a flash of light created at O, a massive particle emitted at O, a piece of paper with a message on it and placed in a bottle. Whatever it is, in order to be present at the other event and hence to either affect it (or even to cause it) this signal must travel the distance Δx in time Δt, i.e. with speed $\Delta x/\Delta t$.

We can now look at what this will mean for each of the events P, Q, R. Firstly, for event P we find from Eq. (137) that $\Delta x/\Delta t > c$. Thus the signal must travel faster than the speed of light, which is not possible. Consequently event O cannot affect, or cause event P. Secondly, for event Q we find from Eq. (138) that $\Delta x/\Delta t < c$ so the signal will travel at a speed less than the speed of light, so event O can affect (or cause) event Q. Finally, for R we find from Eq. (139) that $\Delta x/\Delta t = c$ so that O can effect R by means of a signal travelling at the speed of light. In summary we can write

1. Two events separated by a space-like interval cannot affect one another;
2. Two events separated by a time-like or light-like interval can affect one another.

Thus, returning to our spacetime diagram, we have:

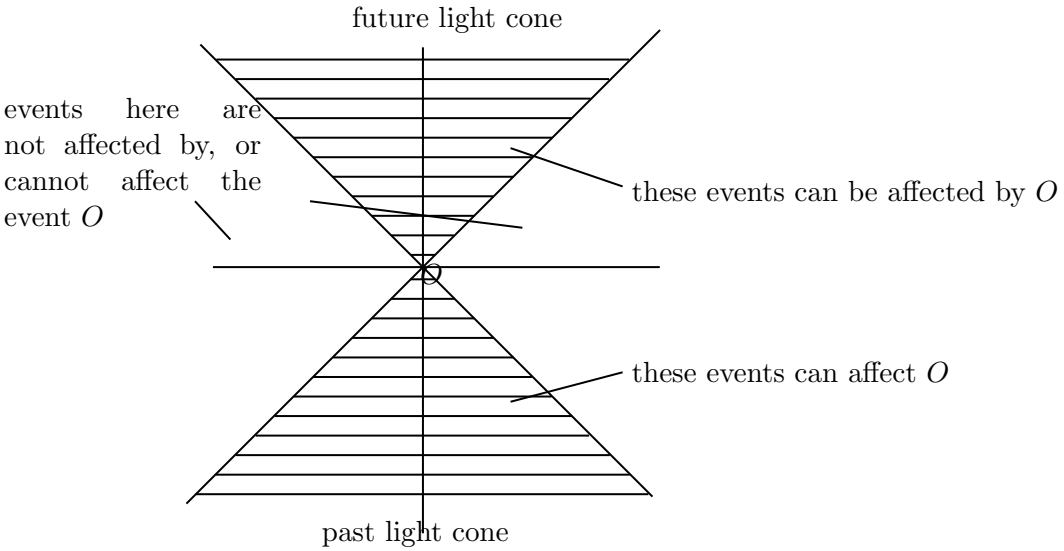

Figure 18: Future and past light cones of the event O

All the events that can be influenced by O constitute the future of event O while all events that can influence O constitute the past of event O.